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New reducible membrane solutions in $D = 11$ supergravity

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Abstract: We construct generalized 11D supergravity solutions of fully localized D2/D6 brane intersections. These solutions are obtained by embedding Taub-NUT and/or self-dual geometries lifted to M-theory. We reduce these solutions to ten dimensions, obtaining new D-brane systems in type-IIA supergravity. We discuss the limits in which the dynamics of the D2 brane decouples from the bulk for these solutions.

Keywords: M-Theory, D-branes, Supergravity Models.

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Contents

1. Introduction

It is believed that fundamental M-theory in the low-energy limit can be described effectively by $D = 11$ supergravity [1]. Extending our understanding of the different classical brane solutions in string theory (or M-theory) is important, and so there is a lot of interest in finding $D = 11$ M-brane solutions such that after reduction to ten dimensions, they (or some combinations of them) reduce simply to the supersymmetric BPS saturated p-brane solutions. Some supersymmetric solutions of two or three orthogonally intersecting 2-branes and 5-branes in $D = 11$ supergravity were obtained some years ago [2], and more such solutions have since been found [3, 4].

Recently an interesting new supergravity solution for a localized D2/D6 intersecting brane system was found [5]. The solution was constructed by lifting a D6 brane to a four-dimensional Taub-NUT geometry embedded in M-theory and then placing M2-branes in the Taub-NUT background geometry.

A special feature of this construction is that the solution is not restricted to be in the near core region of the D6 brane. Moreover, in [6], partially localized D-brane systems involving D3, D4 and D5 branes were constructed. By assuming a simple ansatz for the eleven dimensional metric, the problem reduces to a partial differential equation that is separable and admits proper boundary conditions.

Motivated by this work, we consider extensions of these solutions to more general types of Taub-NUT and/or self-dual geometries lifted to M-theory. We have obtained several different solutions of interest. Specifically, since in the 11 dimensional metric for an M2 brane, the M2 brane itself only takes up two of the 10 spatial coordinates, we can embed a variety of different geometries. These include the Eguchi-Hanson metric, higher-dimensional forms of the Taub-NUT metric, and products of these 4-dimensional metrics. After compactification on a circle, we find the different fields of type-IIA string theory.

In our procedure we begin with a general ansatz for the metric function of an M2 brane in 11-dimensional M-theory. After compactification on a circle (T^1), we find a solution to type IIA theory for which the highest degree of the field strengths is four. Hence the non-compact global symmetry for massless modes is given by the maximal symmetry group $E_{1(1)} = \mathbb{R}$, without any need to dualize the field strengths [7]. For the full type-IIA theory, only the discrete subgroup $E_{1(1)}(\mathbb{Z}) = \mathbb{Z}$ survives, in particular by its action on the BPS spectrum and as a discrete set of identifications on the supergravity moduli space. This subgroup is the U-duality group for all type-IIA theories we find in this paper.

In section 2, we discuss briefly the field equations of supergravity.

Since a natural question to ask is how much supersymmetry is preserved by any brane solution, we divide our solutions into those that preserve some of the supersymmetry (section 3) and those that do not (section 4). We begin by considering some four-dimensional metrics with self-dual curvature. We review the embedding of the Taub-NUT solution in four dimensions (TN_4), including its supersymmetric properties. We obtain a solution dual to that obtained for the D2/D6 intersecting brane system found previously [5]. Another metric with similar self-dual properties is the Eguchi-Hanson (EH) metric, which we consider in section 3.2; we obtain two new D2/D6 intersecting brane solutions for this case. We then discuss embedding products of these metrics: $\mathrm{TN}_4\otimes\mathrm{TN}_4$ (section 3.3), EH$\otimes$EH (section 3.4) and $\mathrm{TN}_4\otimes$EH (section 3.5). All of these solutions preserve some of the supersymmetry, and for completeness we discuss this in an appendix.

We then consider embedding variants of the Taub-NUT spaces. Here we find a rich variety of solutions, though they do not preserve any supersymmetry. In section 4.1, we consider the embedding of four-dimensional Taub-bolt space in M-theory. In this case, we show that the partial differential equation can again be separated into two ordinary differential equations. The solution of one of these differs from the Taub-NUT case [5]; although an analytic solution is not possible, we find a numerical solution of the other equation. In section 4.2, we consider the embedding of six-dimensional Taub-NUT and

Taub-bolt spaces with one NUT charge. Again we find that the partial differential equations can be separated into two ordinary differential equations: one can be solved analytically, and the other one has a proper solution, which we find through numerical methods. In all the above mentioned cases, we find the different type-IIA supergravity NSNS and RR fields. The ten-dimensional metric obtained from reduction of the four-dimensional Taub-bolt is a D2/D6 system. In the case of six-dimensional Taub-NUT(bolt), we find a D2 brane localized in the world-volume of a D4 brane. This solution is a new intersecting brane solution, different from the known intersecting brane obtained by T-dualizing the D-brane solutions [8, 9]. In section 4.3, we consider the embedding of the eight-dimensional Taub-NUT and Taub-bolt spaces with one NUT charge. In these cases, we consider an M-brane solution in a transverse background with one non-compact transverse direction and find exact analytic solutions.

2. General method

We begin with the general lagrangian from which the equations of 11-dimensional supergravity can be constructed [10], briefly reviewing the equations we need here (and employing the same notation). In order to construct a solution to these equations that can be successfully reduced to $D = 10$ dimensional type-IIA string theory, we assume a bosonic ground state, i.e. the VEV of any fermion field should be zero. This will allow us to focus on the equations for g_{MN} and A_{MNP}, which are now given by

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3}\left[F_{MPQR}F_N{}^{PQR} - \frac{1}{8} g_{MN} F_{PQRS}F^{PQRS}\right] \tag{2.1}$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576}\varepsilon^{M_1 \ldots M_8 NPQ} F_{M_1 \ldots M_4} F_{M_5 \ldots M_8} \tag{2.2}$$

where because $\langle \Psi_M \rangle = 0$, F_{MNPQ} is the unmodified four-form field strength

$$\begin{aligned} F_{MNPQ} &= 4\partial_{[M} A_{NPQ]} \\ &= \frac{1}{2}\left[A_{MNP,Q} - A_{NPQ,M} + A_{PQM,N} - A_{QMN,P}\right]. \end{aligned} \tag{2.3}$$

Note that in (2.2) the ε tensor is an 11-component, 11-dimensional tensor, whose usage involves memory-intensive computer algebra calculations. In our case, equation (2.2) is easily shown to be satisfied by all our solutions: due to the non-zero components of the three-form A_{MNP} (or equivalently, non-zero components of the four-form F_{MNPQ}; see eq. (3.2) for more details), the right-hand side of (2.2) is zero, so we need only show that the left-hand side also vanishes.

The general ansatz for an M2 brane solution [11] takes the form

$$ds^2 = H^{1/3}\left[H^{-1}(-dt^2 + dx_1^2 + dx_2^2) + \left(ds^2_{\mathrm{mtrc1}} + ds^2_{\mathrm{mtrc2}}\right)\right] \tag{2.4}$$

where in general H depends on the coordinates transverse to the brane $H = H(x_3, \ldots, x_{10})$, and $A_{tx_1x_2} = 1/H$. We have labelled the eight-dimensional space that is not part of the brane world-volume in (2.4) by two metrics ds^2_{mtrc1} and ds^2_{mtrc2}. We shall take these to be any combination of the following forms: flat space, k-dimensional Taub-NUT/bolt, or Eguchi-Hanson. By embedding a k dimensional (euclideanized) self-dual metric (i.e. a Taub-NUT or Eguchi-Hanson metric) into this equation, we already achieve the form required for Kaluza-Klein reduction of one of the coordinates on a circle. We can then calculate the left-hand side of (2.1) for specific forms of this metric, and using (2.3), calculate the right-hand side and thereby compute H. As in ref. [5], we will only take H to depend on at most two of the transverse coordinates.

From (2.4), one can use

$$g_{AB} = \begin{bmatrix} e^{-2\Phi/3}\left(g_{\alpha\beta} + e^{2\Phi}C_\alpha C_\beta\right) & \nu e^{4\Phi/3}C_\alpha \\ \nu e^{4\Phi/3}C_\beta & \nu^2 e^{4\Phi/3} \end{bmatrix} \tag{2.5}$$

where ν is the winding number, giving the number of times the membrane wraps around the compactified dimension [12]. For simplicity we will take $\nu = 1$ in what follows. From (2.5) and the reduction of A_{MNP} to its ten dimensional form, the Ramond-Ramond (RR) (C_α, $A_{\alpha\beta\gamma}$) and Neveu-Schwarz Neveu-Schwarz (NSNS) (Φ, $B_{\alpha\beta}$ and $g_{\alpha\beta}$) fields can be easily read off. Once the ten dimensional equations are found, their analysis and comparison to existing forms can be carried out. In obtaining the relation (2.5) with $\nu = 1$, we use the well known Kaluza-Klein reduction of the 11D supergravity metric and field strength to 10D metric and field strength [4]

$$ds^2_{(1,10)} = e^{-2\Phi/3}ds^2_{(1,9)} + e^{4\Phi/3}(dx_{10} + C_\alpha dx^\alpha)^2 \tag{2.6}$$

$$F_{(4)} = \mathcal{F}_{(4)} + H_{(3)} \wedge dx_{10} \tag{2.7}$$

where $\mathcal{F}_{(4)}$ and $H_{(3)}$ are the RR four-form and NSNS three-form field strengths corresponding to $A_{\alpha\beta\gamma}$ and $B_{\alpha\beta}$ and x_{10} is the coordinate of compactified manifold. We take it to be a circle with radius R_∞, parameterized as $x_{10} = R_\infty\psi$ where ψ has period 2π. Although we have assumed $\nu = 1$, the $\nu \neq 1$ case can be dealt with by compactifying ν times over this circle and replacing x_{10} by νx_{10} in the relations (2.6) and (2.7). This simply adds to the dilaton field a constant term of the form $\frac{3}{2}\ln\nu$, and multiplies the RR field C_α by a multiplicative constant of $1/\nu$ when we reduce the theory to 10 dimensions.

Preservation of supersymmetry is checked by finding non-trivial solutions to the Killing spinor equation [11]

$$\partial_M\epsilon + \frac{1}{4}\omega_{Mab}\Gamma^{ab}\epsilon + \frac{1}{144}\Gamma_M{}^{NPQR}F_{NPQR}\epsilon - \frac{1}{18}\Gamma^{PQR}F_{MPQR}\epsilon = 0 \tag{2.8}$$

where ϵ is the anticommuting parameter of the supersymmetry transformation and the indices $M, N, \ldots$ are $d = 11$ world indices, while $a, b, \ldots$ are $d = 11$ tangent space indices. Our conventions are

$$de^a = g^a{}_{bc}e^b \wedge e^c \tag{2.9}$$

$$\omega^a{}_{bc} = \frac{1}{2}\left(g^a{}_{bc} + g^b{}_{ca} - g^c{}_{ab}\right) \tag{2.10}$$

$$\omega_{dbM} = \omega^a{}_{bc}\eta_{ad}e^c{}_M \tag{2.11}$$

where the usual definitions and properties

$$e^a = e^a{}_M dx^M \qquad g_{MN} = \eta_{ab} e^a e^b \tag{2.12}$$

(etc.) hold. The Γ's in (2.8) are given by

$$\Gamma^{A_1 \ldots A_p} = \Gamma^{[A_1} \ldots \Gamma^{A_p]} \tag{2.13}$$

and in general we denote $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$. The gamma matrices must also satisfy the Clifford Algebra

$$\left\{\Gamma^a, \Gamma^b\right\} = -2\eta^{ab} \tag{2.14}$$

where we are using the lorentzian signature $[-1, +1, \ldots, +1]$. A representation of the algebra (2.14) is given by

$$\Gamma_{\hat{i}} = \gamma_{\hat{i}} \otimes 1_8 \tag{2.15}$$

and

$$\Gamma_{\xi+4} = \gamma_5 \otimes \widehat{\Gamma}_\xi \tag{2.16}$$

where $\hat{i} = 0, 1, 2, 3$ and $\xi = 0, 1, \ldots, 6$ denotes the spacetime indices for the tangent space groups SO(1,3) and SO(7). The $\Gamma_{\xi+4}$ (and $\widehat{\Gamma}_\xi$) satisfy the anticommutation relations

$$\{\Gamma_{\xi+4}, \Gamma_{\zeta+4}\} = \{\widehat{\Gamma}_\xi\ , \widehat{\Gamma}_\zeta\ \} = -2\delta_{\xi\zeta} \tag{2.17}$$

where the $\widehat{\Gamma}_\xi$'s are given by

$$\begin{aligned}
\widehat{\Gamma}_0 &= i\gamma_0 \otimes 1_2 \\
\widehat{\Gamma}_i &= \gamma_i \otimes 1_2 \\
\widehat{\Gamma}_{i+3} &= i\gamma_5 \otimes \sigma_i
\end{aligned} \tag{2.18}$$

in terms of the Pauli matrices σ_i $(i = 1, 2, 3)$, $\gamma_0 = \left(\begin{smallmatrix} 0 & 1_{16} \\ 1_{16} & 0 \end{smallmatrix}\right)$, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

3. Supersymmetric solutions

3.1 Four dimensional Taub-NUT solution revisited

We review here the embedding of the four-dimensional Taub-NUT solution [5]. The eleven dimensional metric is given by

$$\begin{aligned}
ds_{11}^2 &= H(y,r)^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \\
&\quad + H(y,r)^{1/3}\left(dy^2 + y^2 d\alpha_1^2 + y^2\sin^2(\alpha_1) d\alpha_2^2 + y^2\sin^2(\alpha_1)\sin^2(\alpha_2) d\alpha_3^2 + ds_{TN_4}^2\right)
\end{aligned} \tag{3.1}$$

$$F_{tx_1x_2y} = -\frac{1}{2H^2}\frac{\partial H}{\partial y} \qquad F_{tx_1x_2r} = -\frac{1}{2H^2}\frac{\partial H}{\partial r} \tag{3.2}$$

where $ds_{TN_4}^2$ can be written in either the form given by [5]

$$ds_{TN_4}^2 = \tilde{f}_4(r)\left(dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right) + \left(\frac{(4n)^2}{\tilde{f}_4(r)}\right)\left(d\psi + \frac{1}{2}\cos(\theta)d\phi\right)^2 \tag{3.3}$$

$$\tilde{f}_4(r) = \left(1 + \frac{2n}{r}\right) \tag{3.4}$$

or equivalently as

$$ds^2_{TN_4} = \frac{1}{f_4(r)}\left[d\psi' + 2n\cos(\theta)d\phi\right]^2 + f_4(r)dr^2 + (r^2 - n^2)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{3.5}$$

$$f_4(r) = \frac{(r+n)}{(r-n)}. \tag{3.6}$$

One can go from (3.5) to (3.3) by letting $r \to r + n$ and $\psi' = 4n\psi$.

Requiring that (3.1) and (3.2) satisfy the field equations entails calculating both sides of (2.1) and then solving the resulting differential equation

$$\frac{3(r+2n)}{y}\frac{\partial H(y,r)}{\partial y} + (r+2n)\frac{\partial^2 H(y,r)}{\partial y^2} + 2\frac{\partial H(y,r)}{\partial r} + r\frac{\partial^2 H(y,r)}{\partial r^2} = 0 \tag{3.7}$$

for $H(y,r)$. By substituting

$$H(y,r) = 1 + Q_{M2}Y(y)R(r) \tag{3.8}$$

where Q_{M2} is the M2 brane charge, we obtain

$$\frac{\partial^2 Y}{\partial y^2} + \frac{3}{y}\frac{\partial Y}{\partial y} + p^2 Y = 0 \tag{3.9}$$

whose solution is $J_1(py)/y$. This yields

$$H_{TN_4}(y,r) = 1 + Q_{M2}\int_0^\infty dp \frac{(py)^2 J_1(py)}{4\pi^2 y^3} R_p(r) \tag{3.10}$$

where

$$R_p(r) = \frac{\pi^2 p}{16}\Gamma(pn)\frac{\mathcal{W}_W(-pn, 1/2, 2pr)}{r} = \frac{\pi^2 p^2}{8}\Gamma(pn)e^{-pr}\mathcal{U}(1+pn, 2, 2pr) \tag{3.11}$$

is the solution to

$$r\frac{d^2 R_p(r)}{dr^2} + 2\frac{dR_p(r)}{dr} - p^2\left(r+2n\right)R_p(r) = 0 \tag{3.12}$$

Here $\mathcal{W}_W(x,y,z)$ is the Whittaker-Watson function, related to $\mathcal{U}$, the Kummer U-function or confluent hypergeometric function, as noted above. Reducing (3.1) down to ten dimensions gives rise to the type-IIA supergravity solution of a D2 brane localized along a D6 brane mentioned in the introduction [5].

In addition to the solution (3.10), we easily obtain another solution by changing the separation constant $p \to ic$. In this case the radial equation becomes

$$r\frac{d^2 R_c(r)}{dr^2} + 2\frac{dR_c(r)}{dr} + c^2\left(r+2n\right)R_c(r) = 0 \tag{3.13}$$

and its solution is given by

$$R_c(r) = C_c\frac{(-i)\mathcal{W}_M(-icn, 1/2, 2icr)}{r} = D_c e^{-icr}\,\mathcal{G}(1+icn, 2, 2icr) \tag{3.14}$$

where $\mathcal{G}$ is a hypergeometric function which is finite at $r = 0$, and undergoes damped oscillations until it vanishes at $r = \infty$. Here C_c or D_c are constants that depend only on n and the separation constant c. The solution of the modified differential eq. (3.9) for $Y(y)$ is now

$$Y_c(y) = E_c \frac{K_1(cy)}{y} \tag{3.15}$$

where K_1 is the modified Bessel function, diverging at $y = 0$ and vanishing at infinity. Writing the general solution of the metric function as a superposition of the solutions with constant c, we have

$$\widetilde{H}_{TN_4}(y,r) = 1+Q_{M2}\int_0^\infty dc R_c(r) Y_c(y) = 1+Q_{M2}\int_0^\infty dc f(c) e^{-icr}\mathcal{G}(1+icn,2,2icr)\frac{K_1(cy)}{y} \tag{3.16}$$

where $f(c) = D_c E_c$ and we choose c to belong to the interval $[0,\infty)$. The function $f(c)$ may be determined from a consideration of the near horizon limit. In this limit, where $r \ll n$ (or equivalently when $n \to \infty$ and $r \to 0$), the metric (3.5) reduces to $\mathbb{R}^4$, with the line element

$$ds^2 = dz^2 + z^2 d\Omega_3^2 \tag{3.17}$$

where $z = 2\sqrt{2nr}$. The transverse geometry is then $\mathbb{R}^4 \otimes \mathbb{R}^4$ with the metric

$$ds^2 = dy^2 + dz^2 + (y^2 + z^2) d\Omega_3^2 \tag{3.18}$$

and the metric function (3.16) should coincide with $1 + Q/R^6$, where

$$R = \sqrt{y^2 + z^2} \tag{3.19}$$

Hence we get

$$\begin{aligned}\lim_{z^2 \ll 8n^2} \int_0^\infty dc f(c) e^{-ic\frac{z^2}{8n}} \mathcal{G}\left(1+icn, 2, ic\frac{z^2}{4n}\right)\frac{K_1(cy)}{y} &= \\ = \int_0^\infty dc f(c) \frac{I_1(2ic\sqrt{2nr})}{ic\sqrt{2nr}} \frac{K_1(cy)}{y} &= \frac{1}{(y^2+8nr)^3}\end{aligned} \tag{3.20}$$

which yields $f(c) = c^4/16$, or

$$\widetilde{H}_{TN_4}(y,r) = 1 + \frac{Q_{M2}}{16}\int_0^\infty dc c^4 e^{-icr}\mathcal{G}(1+icn,2,2icr)\frac{K_1(cy)}{y} \tag{3.21}$$

This same approach can be used to determine the form of the integral in equation (3.10), though we note that a different method was employed in ref. [5]. In figure 1, the log-log plots of $h(r) \simeq (H_{TN_4}(y=0,r)-1)$ versus r/n and $\widetilde{h}(y) \simeq (\widetilde{H}_{TN_4}(y,r=0)-1)$ versus y/n, are given where we choose the normalization coefficients such that two functions approach one as r and y go to zero. The two functions have the same behaviour qualitatively, though $\widetilde{h}(y)$ has a considerably steeper falloff.

By inserting either of the solutions (3.10) or (3.21) into the metric (3.1), it is straightforward to show that both solutions preserve 1/4 of the supersymmetry, as we demonstrate in appendix A.

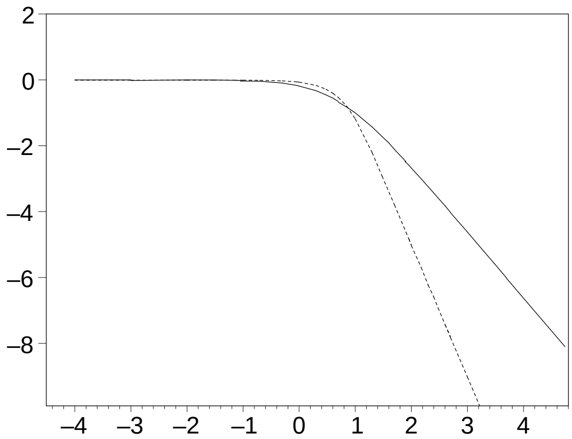

Figure 1: Log-Log plots of the functions $h(r)$ (solid) and $\widetilde{h}(y)$ (dotted) in terms of r/n and y/n, respectively.

Dimensional reduction of (3.1) with (3.10) or (3.21) along the coordinate ψ of the Taub-NUT$_4$ metric gives the type-IIA supergravity metric

$$\begin{aligned} ds_{10}^2 &= H^{-1/2}\widetilde{f}_4^{-1/2}\left(-dt^2+dx_1^2+dx_2^2\right)+ \\ &\quad +H^{1/2}\widetilde{f}_4^{-1/2}\left(dy^2+y^2d\Omega_3^2\right)+H^{1/2}\widetilde{f}_4^{1/2}(dr^2+r^2d\Omega_2^2) \end{aligned} \tag{3.22}$$

which describes a D2 brane localized along the world-volume of D6 brane. The other fields in ten dimensions are NSNS fields

$$\Phi = \frac{3}{4}\ln\left\{\frac{H^{1/3}}{\widetilde{f}_4}\right\} \tag{3.23}$$

$$B_{\mu\nu} = 0 \tag{3.24}$$

and Ramond-Ramond (RR) fields

$$C_\phi = 2n\cos(\theta) \tag{3.25}$$

$$A_{tx_1x_2} = \frac{1}{H} \tag{3.26}$$

where in the above relations, H refers either to $H_{TN_4}(y,r)$ or $\widetilde{H}_{TN_4}(y,r)$ given by (3.10) and (3.21).

3.2 4D Eguchi-Hanson

The Eguchi-Hanson metric is another asymptotically flat metric with self-dual curvature that can also be embedded into eleven dimensions, furnishing a localized brane solution similar to the Taub-NUT metric above. The eleven dimensional metric will be the same as (3.1), but with $ds_{TN_4}^2$ replaced with [13]

$$ds_{EH}^2 = \frac{r^2}{4g(r)}\left[d\psi+\cos(\theta)d\phi\right]^2+g(r)dr^2+\frac{r^2}{4}\left(d\theta^2+\sin^2(\theta)d\phi^2\right) \tag{3.27}$$

$$g(r) = \left(1-\frac{a^4}{r^4}\right)^{-1}. \tag{3.28}$$

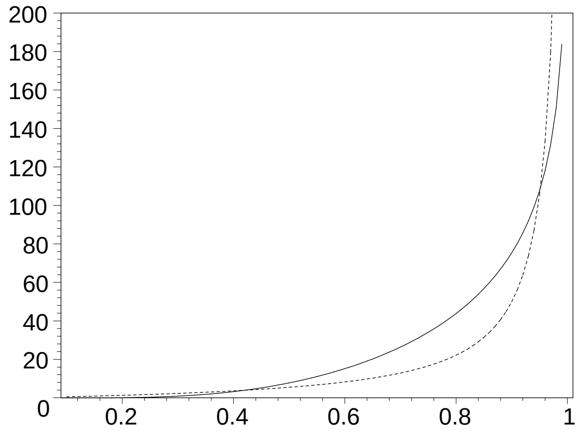

Figure 2: Numerical solutions $R_c/10^{48}$ of the radial equation (3.30) as a function of a/r (solid), and solution $R_p/10^6$ of the radial equation (3.12) for TN_4 as a function of n/r (dotted). For $r \approx a$ (or n), both R's diverge and for $r \approx \infty$, they vanish.

The four-form field strength is given again by (3.2). We find that this new eleven dimensional metric will satisfy the supergravity field equations, provided

$$\frac{(3r^4 + a^4)}{r^5}\frac{\partial H}{\partial r} + \frac{(r-a)(r+a)(r^2+a^2)}{r^4}\frac{\partial^2 H}{\partial r^2} + \frac{3}{y}\frac{\partial H}{\partial y} + \frac{\partial^2 H}{\partial y^2} = 0\,. \tag{3.29}$$

This equation is separable, and becomes

$$\frac{(r^4 - a^4)}{r^4}\frac{d^2 R_c(r)}{dr^2} + \frac{(3r^4 + a^4)}{r^5}\frac{dR_c(r)}{dr} - c^2 R_c(r) = 0 \tag{3.30}$$

$$\frac{d^2 Y(y)}{dy^2} + \frac{3}{y}\frac{dY(y)}{dy} + c^2 Y(y) = 0 \tag{3.31}$$

upon substituting in $H(y,r) = 1 + Q_{M2} Y(y) R_c(r)$, where c^2 is a separation constant. Eq. (3.31) is identical to equation (3.9) and so we again obtain

$$Y(y) \sim \frac{J_1(cy)}{y}\,. \tag{3.32}$$

We can solve the equation for $R_c(r)$ numerically. For large r, the solution of the equation (3.30) that vanishes at infinity is $K_1(cr)/r$, where K_1 is the modified Bessel function. A typical numerical solution to the radial differential equation (3.30) versus a/r is displayed in figure 2. Note that in the metric (3.27), the coordinate r must be greater than or equal to a. For comparison, a plot of the $R_p(r)$ given by the relation (3.11) is also plotted in figure 2; qualitatively they are the same, though the EH function has a less rapid falloff. Roughly speaking, the Eguchi-Hanson parameter a plays the same role of the NUT charge n in the TN_4 case.

We obtain

$$H(y,r) = 1 + Q_{M2}\int_0^\infty dc\, g(c)\frac{J_1(cy)}{y} R_c(r)\,. \tag{3.33}$$

To fix the measure function $g(c)$, we compare the above relation to that of a metric function of a brane in an 8-dimensional flat metric $\mathbb{R}^4 \otimes \mathbb{R}^2 \otimes S^2$, obtained by looking at the near horizon limit. We note that for $r = a(1+\epsilon^2)$, where $\epsilon \ll 1$, the metric reduces to

$$ds^2_{r=a(1+\epsilon^2)} = a^2 \left\{ \epsilon^2 \left[d\psi + \cos(\theta) d\phi \right]^2 + d\epsilon^2 \right\} + \frac{a^2}{4} \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \tag{3.34}$$

$$\approx z^2 d\psi^2 + dz^2 + \frac{a^2}{4} \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \tag{3.35}$$

which is $\mathbb{R}^2 \otimes S^2$ with the radial length equal to $\sqrt{z^2 + a^2/4}$. If we assume that the parameter a is small, the differential equation (3.30) reduces to

$$\ddot{R}_c + \frac{3}{\widehat{r}} \dot{R}_c - c^2 R_c = 0 \tag{3.36}$$

where $\widehat{r} = r - a = a\epsilon^2 \approx 0$ and the overdot denotes $d/d\widehat{r}$. This equation has the solution

$$R_c(\widehat{r}) = \frac{A_c}{\widehat{r}} K_1(c\widehat{r}) + \frac{B_c}{\widehat{r}} I_1(c\widehat{r}) \tag{3.37}$$

which will vanish at infinity provided $B_c = 0$, though it will diverge at $\widehat{r} = 0$.

Taking these limits into account in equation (3.33), we find

$$\int_0^\infty dc g(c) \frac{K_1(c\widehat{r})}{\widehat{r}} J_1(cy) = \lim_{a \to 0} \frac{y}{(z^2 + \frac{a^2}{4} + y^2)^3}$$

$$= \lim_{a \to 0} \frac{y}{(\widehat{r}a + y^2)^3} = \frac{1}{y^5} \tag{3.38}$$

where we can absorb the constant A_c into the definition of $g(c)$. By comparing the above relation with the known integral

$$\int_0^\infty dc c^3 \frac{K_1(c\widehat{r})}{\widehat{r}} J_1(cy) = \frac{8y}{(\widehat{r}^2 + y^2)^3} \overset{\widehat{r}=a\epsilon^2}{\to} \frac{8}{y^5} \tag{3.39}$$

we find that $g(c) = c^4/8$, and so the metric function becomes

$$H_{EH}(y,r) = 1 + Q_{M2} \int_0^\infty dc \left(\frac{c^4}{8} \right) R_c(r) \frac{J_1(cy)}{y} . \tag{3.40}$$

By changing $c \to ic$ in the differential equations (3.30) and (3.31), we get another solution in the form of

$$\widetilde{H}_{EH}(y,r) = 1 + Q_{M2} \int_0^\infty dc \left(\frac{c^4}{8} \right) \widetilde{R}_c(r) \frac{K_1(cy)}{y} \tag{3.41}$$

where $\widetilde{R}_c(r)$ is a damped oscillating function whose behaviour is plotted in figure 3 and $\widetilde{g}(c) = c^4/8$.

Equations (3.40) and (3.41) are the respective analogues of the TN_4 solutions (3.10) and (3.21) for the EH case. They can be plotted numerically to yield results qualitatively similar to those obtained for H_{TN4} and $\widetilde{H}_{TN4}$ in the previous section; we will not present them here.

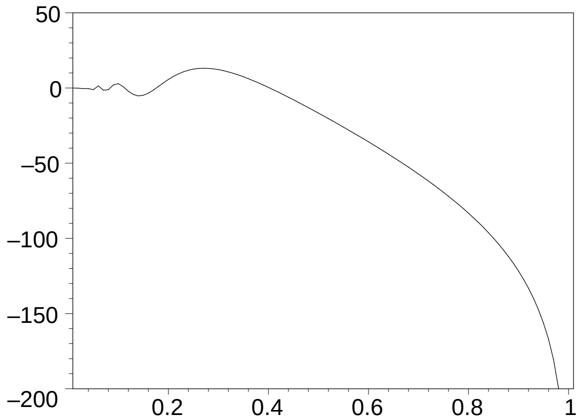

Figure 3: Numerical solution of radial equation (3.68) for $\mathcal{R}_2/10^5$ (or $\widetilde{R}_c/10^5$) as a function of $1/r_2$ (or $1/r$) for non-zero separation constant. The Eguchi-Hanson parameters a_2 (a) are set to one and so for $r_2 \approx a_2$ ($r \approx a$), the function $\mathcal{R}_2$ ($\widetilde{R}_c$) diverges and for $r_2 \approx \infty$ ($r \approx \infty$), it vanishes.

Reduction of these solutions to a ten dimensional type-IIA string theory solution proceeds in a manner similar to the reduction of S^4 regarded as containing a NUT and anti-NUT charge [14]. The vector $\partial/\partial\psi$ generates a Hopf fibration of a 3-sphere in the metric (3.27). Using (2.5) we obtain the NSNS fields

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left\{\frac{H^{1/3}w^2}{4g}\right\} \\ B_{\mu\nu} &= 0 \end{aligned} \tag{3.42}$$

where we define the dimensionless coordinate w by $w = r/a$. The Ramond-Ramond (RR) fields and the ten dimensional metric will be given by

$$\begin{aligned} C_\phi &= a\cos(\theta) \\ A_{tx_1x_2} &= \frac{1}{H} \end{aligned} \tag{3.43}$$

$$\begin{aligned} ds^2_{10} = \frac{w}{2}\Big\{ & H^{-1/2}g^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H^{1/2}g^{-1/2}\left(dy^2 + y^2 d\Omega_3^2\right) + \\ & + H^{1/2}g^{1/2}a^2\left(dw^2 + \frac{w^2}{4g}d\Omega_2^2\right)\Big\}\,. \end{aligned} \tag{3.44}$$

The metric (3.44) describes a D2/D6 system where the D2-brane is localized along the world-volume of the D6-brane. We note H refers to either H_{EH} or $\widetilde{H}_{EH}$ and $g = g(w) = \left(1 - w^{-4}\right)^{-1}$ in the above relations. We have explicitly checked that the above 10-dimensional metric, with the given dilaton and one form, is a solution to the 10-dimensional supergravity equations of motion.

The metric (3.44) is locally asymptotically flat (though the dilaton field diverges); for large w it reduces to

$$ds^2_{10} = \frac{w}{2}\left\{-dt^2 + dx_1^2 + dx_2^2 + dy^2 + y^2 d\Omega_3^2 + a^2\left(dw^2 + \frac{w^2}{4}d\Omega_2^2\right)\right\} \tag{3.45}$$

which is a 10D locally flat metric with solid deficit angles. The Kretchmann invariant of this spacetime vanishes at infinity and is given by

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{224}{a^4 w^6} \tag{3.46}$$

and all the components of the Riemann tensor in the orthonormal basis have similar $1/w^3$ behaviour, vanishing at infinity.

To calculate how much supersymmetry is preserved by this solution in eleven dimensions, we again use the spinor equation (2.8). Half of the supersymmetry is again removed via the projection operator $(1+\Gamma^{\hat{t}\hat{x}_1\hat{x}_2})\epsilon = 0$, due to the presence of the brane, and another half is removed due to the self-dual nature of the EH metric, as shown in appendix A. Hence embedding the EH metric into an eleven dimensional M2 brane metric preserves 1/4 of the supersymmetry (see appendix A for details).

3.3 Taub-NUT$_4$ $\otimes$ Taub-NUT$_4$

We can also embed two four dimensional metrics into the eleven dimensional membrane metric. The first case we consider here is embedding two TN$_4$ metrics, giving the following metric:

$$ds_{11}^2 = H(r_1,r_2)^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H(r_1,r_2)^{1/3}\left(ds^2_{TN_4(1)} + ds^2_{TN_4(2)}\right) \tag{3.47}$$

$$ds^2_{TN_4(i)} = \left(\frac{(4n_i)^2}{\widetilde{f}_i(r_i)}\right)\left(d\psi_i + \frac{1}{2}\cos(\theta_i)d\phi_i\right)^2 + \widetilde{f}_i(r_i)\left(dr_i^2 + r_i^2\left(d\theta_i^2 + \sin^2(\theta_i)d\phi_i^2\right)\right) \tag{3.48}$$

$$F_{tx_1x_2r_i} = -\frac{1}{2H^2}\frac{\partial H}{\partial r_i} \tag{3.49}$$

with $\widetilde{f}_i(r_i) = (1 + 2n_i/r_i)$. Note that we can also use the (3.5) form for the Taub-NUT metrics (which is easier for the supersymmetry check), but for reduction down to ten dimensions the form given in equation (3.3) is preferable, as ψ has a period of 2π.

We can choose to compactify down to ten dimensions by compactifying on either ψ_1 or ψ_2. Since either way produces the same results (with $1 \leftrightarrow 2$), we will compactify on ψ_2. Thus, the radius R_∞ of the circle as $r \to \infty$ with line element $R_\infty\left(d\psi_2 + \frac{1}{2}\cos\theta_2 d\phi_2\right)$ is the same as in the Taub-NUT case above,

$$R_\infty = 4n_2\,. \tag{3.50}$$

Note that the r_i range from $[0,\infty)$, the ψ_i and the ϕ_i range over $[0, 2\pi]$, and the θ_i range over $[0,\pi]$.

The metric (3.47) and (3.49) satisfies the eleven dimensional equations of motion if the harmonic function satisfies the differential equation

$$\begin{aligned} &2(r_2+2n_2)\frac{\partial H(r_1,r_2)}{\partial r_1} + r_1(r_2+2n_2)\frac{\partial^2 H(r_1,r_2)}{\partial r_1^2} + \\ &+ 2(r_1+2n_1)\frac{\partial H(r_1,r_2)}{\partial r_2} + r_2(r_1+2n_1)\frac{\partial^2 H(r_1,r_2)}{\partial r_2^2} = 0\,. \end{aligned} \tag{3.51}$$

This equation is separable, which can again be seen by substituting in $H(r_1, r_2) = 1 + Q_{M2}R_1(r_1)R_2(r_2)$. This gives two equations of the same form

$$r_i \frac{\partial^2 R_i(r_i)}{\partial r_i^2} + 2\frac{\partial R_i(r_i)}{\partial r_i} \pm c^2(r_i + 2n_i)R_i(r_i) = 0 \tag{3.52}$$

where the $\pm c^2$ term is present because the separable equations must be set equal to a constant (c^2) of the opposite sign in order for (3.51) to equal zero. The most general solution is the product of the exponentially decayed and damped oscillating functions discussed in section 3.1, yielding

$$\begin{aligned} H &= 1 + Q_{M2}\int_0^\infty dc R_{1c}(r_1)R_{2c}(r_2) \\ &= 1 + Q_{M2}\int_0^\infty dc\, h(c)e^{-icr_1}\mathcal{G}(1 + icn_1, 2, 2icr_1)e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) \end{aligned} \tag{3.53}$$

The metric of the transverse geometry in the near horizon limit reduces to $R^4 \otimes R^4$, with the metric

$$ds^2 = dz_1{}^2 + dz_2^2 + z_1^2 d\Omega_3^2 + z_2^2 d\Omega_3'^2 \tag{3.54}$$

where $z_1 = 2\sqrt{2n_1r_1}, z_2 = 2\sqrt{2n_2r_2}$. So we should have

$$\lim_{r_1 \ll n_1 \text{ and } r_2 \ll n_2} \int_0^\infty dch(c)e^{-icr_1}\mathcal{G}(1 + icn_1, 2, 2icr_1)e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) = \frac{1}{(z_1^2 + z_2^2)^3}\,. \tag{3.55}$$

Using the relations for the limiting values of the hypergeometric and confluent hypergeometric functions [15], the above relation becomes

$$\int_0^\infty dch(c)\frac{I_1(2ic\sqrt{2n_1r_1})}{c}\frac{2K_1(2c\sqrt{2n_2r_2})}{c\Gamma(cn_2)} = \frac{i\sqrt{2n_1r_1}\sqrt{2n_2r_2}}{512(n_1r_1 + n_2r_2)^3} \tag{3.56}$$

which yields $h(c) = \frac{c^5}{64}\Gamma(cn_2)$ so that

$$\begin{aligned} H &= H_{(TN)^2}(r_1, r_2) \\ &= 1 + Q_{M2}\int_0^\infty dc\frac{c^5}{64}\Gamma(cn_2)e^{-icr_1}\mathcal{G}(1 + icn_1, 2, 2icr_1)e^{-cr_2}\mathcal{U}(1 + cn_2, 2, 2cr_2) \end{aligned} \tag{3.57}$$

Note that changing $c \to ic$ in (3.52) does not yield any other solution, but instead interchanges the functions $R_1 \longleftrightarrow R_2$. The graph of the metric function $H_{(TN)^2}(r_1 = 0, r_2) - 1$, is exactly the same as the graph of the function $h(r)$ in the figure 1.

Next we wish to dimensionally reduce the solution to the type-IIA string theory in ten dimensions. Doing this will give the NSNS fields

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left(\frac{H^{1/3}}{\widetilde{f}_2}\right) \\ B_{\mu\nu} &= 0 \end{aligned} \tag{3.58}$$

and RR fields

$$
\begin{aligned}
C_{\phi_2} &= 2n_2\cos(\theta_2) \\
A_{tx_1x_2} &= H^{-1}\,.
\end{aligned}
\tag{3.59}
$$

The metric in ten dimensions will be given by

$$
\begin{aligned}
ds^2_{10} = {} & H(r_1,r_2)^{-1/2}\widetilde{f}_2^{-1/2}\left(-dt^2+dx_1^2+dx_2^2\right)+H(r_1,r_2)^{1/2}\widetilde{f}_2^{-1/2}\left(ds^2_{TN_4(1)}\right)+ \\
& +H(r_1,r_2)^{1/2}\widetilde{f}_2^{1/2}\left(dr_2^2+r_2^2\left(d\theta_2^2+\sin^2(\theta_2)d\phi_2^2\right)\right)
\end{aligned}
\tag{3.60}
$$

This represents a D2/D6 brane system, where $H = H_{(TN)^2}$ in the preceding equation (3.57).

We can further reduce the metric (3.60) along the ψ_1 direction of the $TN_4(1)$. However the result of this compactification is not the same as the reduction of the M-theory solution (3.47) over a torus, which is compactified type-IIB theory. The reason is that to get the compactified type-IIB theory, we should compactify the T-dual of the IIA metric (3.60) over a circle, and not directly compactify the 10D IIA metric (3.60) along the ψ_1 direction.

We note also an interesting result in reducing the 11D metric (3.47) along the ψ_1 (or ψ_2) direction of the $TN_4(1)$ (or $TN_4(2)$) in large radial coordinates. As r_1 (or r_2) $\to\infty$ the transverse geometry in (3.47) locally approaches $\mathbb{R}^3\otimes S^1\otimes TN_4(2)$ (or $TN_4(1)\otimes\mathbb{R}^3\otimes S^1$) and so the reduced theory, obtained by compactification over the circle of the TN_4, is IIA. Then by T-dualization of this theory (on the remaining S^1 of the transverse geometry), we find a type-IIB theory which describes the D5 defects.

This solution also preserves 1/4 of the supersymmetry. The gauge terms and partial derivatives of $H(r_1,r_2)$ will remove half the supersymmetry through the required use of (A.8). After this, we get two sets of equations of the form (A.12), (A.13) and (A.14) (with $(\psi,\theta,\phi)\to(\psi_i,\theta_i,\phi_i)$, $i=1,2$). These can be solved by use of the equivalent of (A.16);

$$
\Gamma^{\hat{\psi}_1\hat{r}_1\hat{\theta}_1\hat{\phi}_1}\epsilon = \epsilon \tag{3.61}
$$

$$
\Gamma^{\hat{\psi}_2\hat{r}_2\hat{\theta}_2\hat{\phi}_2}\epsilon = \epsilon \tag{3.62}
$$

However, due to the fact that we're in eleven dimensions, (A.8) and (3.61) imply (3.62), and so we actually only need two projection operators, not three, meaning 1/4 of the supersymmetry is preserved.

3.4 Eguchi-Hanson $\otimes$ Eguchi-Hanson

We can also embed two Eguchi-Hanson metrics into eleven dimensions;

$$
ds^2_{11} = H(r_1,r_2)^{-2/3}\left(-dt^2+dx_1^2+dx_2^2\right)+H(r_1,r_2)^{1/3}\left(ds^2_{EH1}+ds^2_{EH2}\right) \tag{3.63}
$$

$$
ds^2_{EH(i)} = \frac{r_i^2}{4g_i(r_i)}\left[d\psi_i+\cos(\theta_i)d\phi_i\right]^2+g_i(r_i)dr_i^2+\frac{r_i^2}{4}\left(d\theta_i^2+\sin^2(\theta_i)d\phi_i^2\right) \tag{3.64}
$$

$$
g_i(r_i) = \left(1-\frac{a_i^4}{r_i^4}\right)^{-1}, \qquad F_{tx_1x_2r_i} = -\frac{1}{2H^2}\frac{\partial H}{\partial r_i}\,. \tag{3.65}
$$

This metric will satisfy the supergravity equations if the following differential equation holds;

$$\frac{(3r_1^4+a_1^4)}{r_1^5}\frac{\partial H(r_1,r_2)}{\partial r_1}+\frac{(r_1-a_1)(r_1+a_1)(r_1^2+a_1^2)}{r_1^4}\frac{\partial^2 H(r_1,r_2)}{\partial r_1^2}+$$
$$+\frac{(a_2^4+3r_2^4)}{r_2^5}\frac{\partial H(r_1,r_2)}{\partial r_2}+\frac{(r_2-a_2)(r_2+a_2)(r_2^2+a_2^2)}{r_2^4}\frac{\partial^2 H(r_1,r_2)}{\partial r_2^2}=0 \tag{3.66}$$

This is again separable, using $H(r_1,r_2)=1+Q_{M2}\mathcal{R}_1(r_1)\mathcal{R}_2(r_2)$, and gives the following two differential equations,

$$r_1(r_1^4-a_1^4)\frac{d^2\mathcal{R}_1(r_1)}{dr_1^2}+(3r_1^4+a_1^4)\frac{d\mathcal{R}_1(r_1)}{dr_1}-\mathcal{R}_1(r_1)r_1^5c^2=0 \tag{3.67}$$
$$r_2(r_2^4-a_2^4)\frac{d^2\mathcal{R}_2(r_2)}{dr_2^2}+(3r_2^4+a_2^4)\frac{d\mathcal{R}_2(r_2)}{dr_2}+\mathcal{R}_2(r_2)r_2^5c^2=0\,. \tag{3.68}$$

These equations are the same as the $R_c(r)$ equation in (3.30). Like that equation, these are not solvable analytically (unless $c=0$, which reduces generality), but are solvable numerically. Since equation (3.67) is the same as the radial differential equation (3.30), the numerical solution of this equation given in figure 2. It vanishes at large r_1 and increases monotonically with increasing $1/r_1$, diverging logarithmically at $r_1\approx a_1$.

The solution of the other differential equation (3.68) has damped oscillating behaviour and diverges at $r_2\approx a_2$. A typical solution for $\mathcal{R}_2(r_2)$, is given in figure 3.

The most general solution is therefore given by

$$H=1+Q_{M2}\int_0^\infty dc\, j(c)\mathcal{R}_{1c}(r_1)\mathcal{R}_{2c}(r_2)\,. \tag{3.69}$$

The transverse geometry in the limit of small a_1 and a_2 reduces to $\mathbb{R}^2\otimes S^2\otimes\mathbb{R}^2\otimes S^2$ with the radial length equal to $\sqrt{z_1^2+z_2^2+\frac{a_1^2}{4}+\frac{a_2^2}{4}}$. In this limit, the differential equation (3.67) has the solution (3.37) and (3.68) has the solution

$$\mathcal{R}_{2c}(\widehat{r}_2)=\frac{A_{2c}}{\widehat{r}_2}J_1(c\widehat{r}_2)+\frac{B_{2c}}{\widehat{r}_2}Y_1(c\widehat{r}_2) \tag{3.70}$$

which is finite at $\widehat{r}_2=0$ and vanishes at infinity provided $B_{2c}=0$.

Taking these limits into account in equation (3.69), we must have

$$\begin{aligned}\int_0^\infty dc\frac{j(c)}{\widehat{r}_1\widehat{r}_2}K_1(c\widehat{r}_1)J_1(c\widehat{r}_2) &= \lim_{a_1,a_2\to 0}\frac{1}{(z_1^2+z_2^2+\frac{a_1^2}{4}+\frac{a_2^2}{4})^3}\\ &= \frac{1}{(\widehat{r}_1a_1+\widehat{r}_2a_2)^3}\end{aligned} \tag{3.71}$$

where we absorb the constants A_{1c},A_{2c} into the definition of $j(c)$. By comparing the above relation with the known integral

$$\int_0^\infty dc c^3K_1(c\widehat{r}_1)J_1(c\widehat{r}_2)=\frac{8\widehat{r}_1\widehat{r}_2}{(\widehat{r}_1^2+\widehat{r}_2^2)^3}\overset{a_1,a_2\to 0}{\sim}\frac{8\widehat{r}_1\widehat{r}_2}{(\widehat{r}_1a_1+\widehat{r}_2a_2)^3} \tag{3.72}$$

we find that $j(c) = c^5/8$ and the metric function becomes

$$H = H_{(EH)^2}(r_1, r_2) = 1 + Q_{M2} \int_0^\infty dc \left\{ \frac{c^5}{8} \right\} \mathcal{R}_{1c}(r_1) \mathcal{R}_{2c}(r_2) \,. \tag{3.73}$$

As before, changing $c \to ic$ in (3.67) and (3.68) merely interchanges the functions $\mathcal{R}_1 \longleftrightarrow \mathcal{R}_2$ and yields no new solutions.

Next we reduce this metric down to ten dimensions. As with the $\mathrm{TN}_4 \otimes \mathrm{TN}_4$ case above, we can chose to compactify on either ψ_1 or ψ_2, without loss of generality, so again we choose the ψ_2 coordinate. Upon compactification, we get the same NSNS and RR fields as in the relations (3.42) and (3.43) in section 3.2 (with $(w, \theta, \phi, g, a) \to (w_2, \theta_2, \phi_2, g_2, a_2)$), and the metric becomes

$$\begin{aligned} ds_{10}^2 = \frac{w_2}{2} \Big\{ & H^{-1/2} g_2^{-1/2} \left(-dt^2 + dx_1^2 + dx_2^2\right) + \\ & + H^{1/2} g_2^{-1/2} \left(ds^2_{EH(1)}\right) + H^{1/2} g_2^{1/2} a_2^2 \left(dw_2^2 + \frac{w_2^2}{4 g_2} \left(d\theta_2^2 + \sin^2(\theta_2) d\phi_2^2\right)\right)\Big\} \end{aligned} \tag{3.74}$$

which describes another D2/D6 system, where $H = H_{(EH)^2}$. In the large w_2 limit ($w_2 = r_2/a_2$), the metric (3.74), reduces to

$$ds_{10}^2 = \frac{w_2}{2} \left\{ -dt^2 + dx_1^2 + dx_2^2 + ds^2_{EH(1)} + dr_2^2 + a_2^2 \frac{w_2^2}{4} \left(d\theta_2^2 + \sin^2(\theta_2) d\phi_2^2\right) \right\} \tag{3.75}$$

which is a 10D locally asymptotically flat metric. The Kretchmann invariant of this space-time is given by

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{32(48 a_1^8 a_2^4 w_2^4 + 7 r_1^{12})}{r_1^{12} a_2^4 w_2^6} = \left(1536 \frac{a_1^8}{r_1^{12}}\right) \frac{1}{w_2^2} + \frac{224}{a_2^4} \frac{1}{w_2^6} \tag{3.76}$$

which vanishes at large w_2, (though not as rapidly as the case with only one EH subspace) and all the components of the Riemann tensor in an orthonormal basis approach zero as $w_2 \to \infty$.

This solution also preserves 1/4 of the supersymmetry (and not the 1/8 one might expect), for the same reason as in section 3.3. The projection operator (A.8) can be used to deal with the gauge terms, and we're left with two sets of equations of the form (A.18), (A.19) and (A.20) (with $\psi, r, \theta, \phi \to \psi_i, r_i, \theta_i, \phi_i,\ i = 1, 2$), which can be dealt with by two more projection operators;

$$\Gamma^{\hat\psi_1 \hat r_1 \hat\theta_1 \hat\phi_1} \epsilon = -\epsilon \tag{3.77}$$

$$\Gamma^{\hat\psi_2 \hat r_2 \hat\theta_2 \hat\phi_2} \epsilon = -\epsilon \tag{3.78}$$

and as before, in eleven dimensions two of the projection operators, say (A.8) and (3.77), imply the third.

3.5 Taub-NUT$_4$ ⊗ Eguchi-Hanson

Finally, we can also have a Taub-NUT$_4$ and an Eguchi-Hanson metric together embedded into eleven dimensions, giving

$$ds_{11}^2 = H(r_1, r_2)^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H(r_1, r_2)^{1/3}\left(ds^2_{TN_4(1)} + ds^2_{EH(2)}\right) \tag{3.79}$$

with $ds^2_{TN_4(1)}$ and $ds^2_{EH(2)}$ given by (3.48) ($i = 1$) and (3.64) ($i = 2$). This is a solution to the supergravity equations provided the differential equation

$$\begin{aligned}\frac{2}{(r_1 + 2n_1)}\frac{\partial H}{\partial r_1} + \frac{r_1}{(r_1 + 2n_1)}\frac{\partial^2 H}{\partial r_1^2} +& \\ +\frac{(a_2^4 + 3r_2^4)}{r_2^5}\frac{\partial H}{\partial r_2} + \frac{(r_2 - a_2)(r_2 + a_2)(r_2^2 + a_2^2)}{r_2^4}\frac{\partial^2 H}{\partial r_2^2} &= 0\end{aligned} \tag{3.80}$$

is satisfied. This is separable using $H(r_1, r_2) = 1 + Q_{M2}R(r_1)\mathcal{R}(r_2)$. The result is two differential equations, one being (3.52) with ($i = 1$) and the other being (3.68). The most general solution is given by

$$H_A(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dck(c)e^{-cr_1}\mathcal{U}(1 + n_1 c, 2, 2cr_1)\mathcal{R}_2(r_2) \tag{3.81}$$

where we have chosen the separation constant so that the TN solution exponentially decays. To determine the function $k(c)$ we consider the special case $n_1 \to \infty$ and $a_2 \to 0$, where the radial length in the transverse space to M2 brane is equal to $\sqrt{z_1^2 + z_2^2 + a_2^2/4}$, yielding

$$\int_0^\infty dck(c)\frac{2K_1(2c\sqrt{2n_1 r_1})}{c\Gamma(cn_1)}\frac{A_{2c}}{\widehat{r}_2}J_1(c\widehat{r}_2) = \frac{\sqrt{2n_1 r_1}}{(z_1^2 + z_2^2 + \frac{a_2^2}{4})^3} \tag{3.82}$$

which gives $k(c) = \frac{c^5}{16}\Gamma(cn_1)$, and so in turn

$$H_A(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dc\frac{c^5}{16}\Gamma(cn_1)e^{-cr_1}\mathcal{U}(1 + n_1 c, 2, 2cr_1)\mathcal{R}_2(r_2)\,. \tag{3.83}$$

The alternate case is

$$H_B(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dcl(c)e^{-icr_1}\mathcal{G}(1 + in_1 c, 2, 2icr_1)\mathcal{R}_1(r_2) \tag{3.84}$$

where $\mathcal{R}_1(r_2)$ is the solution of EH radial differential equation (3.67), which decays at large r_2, (refer to figure 2). The function $l(c) = c^5/8$ by similar methods and so

$$H_B(r_1, r_2) = 1 + Q_{M2}\int_0^\infty dc\frac{c^5}{8}e^{-icr_1}\mathcal{G}(1 + in_1 c, 2, 2icr_1)\mathcal{R}_1(r_2)\,. \tag{3.85}$$

It can of course be shown that this solution preserves 1/4 of the supersymmetry, by following the same reasoning that shows the Taub-NUT ⊗ Taub-NUT or Eguchi-Hanson ⊗ Eguchi-Hanson solutions preserve 1/4 of the supersymmetry.

For this solution, we can reduce along either the ψ_1 or ψ_2 directions, giving two different ten dimensional metrics.

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3.5.1 Case 1: reduction along ψ_1

Reducing along the ψ_1 direction from the Taub-NUT metric gives the following fields

$$\Phi = \frac{3}{4}\ln\left(\frac{H^{1/3}}{\widetilde{f}_1}\right) \tag{3.86}$$

$$B_{\mu\nu} = 0 \tag{3.87}$$

$$C_{\phi_1} = 2n_1\cos(\theta_1) \tag{3.88}$$

$$A_{tx_1x_2} = H^{-1} \tag{3.89}$$

and gives the following metrics

$$\begin{aligned} ds_{10}^2 = & H^{-1/2}\widetilde{f}_1^{-1/2}\left(-dt^2+dx_1^2+dx_2^2\right)+H^{1/2}\widetilde{f}_1^{-1/2}ds^2_{EH(2)}+ \\ & +H^{1/2}\widetilde{f}_1^{1/2}\left(dr_1^2+r_1^2\left(d\theta_1^2+\sin^2(\theta_1)d\phi_1^2\right)\right) \end{aligned} \tag{3.90}$$

which describes two D2/D6 systems (respectively corresponding to H equalling $H_A(r_1,r_2)$ or $H_B(r_1,r_2)$), where the D2-brane is localized along the world-volume of the other one.

3.5.2 Case 2: reduction along ψ_2

We can also reduce along the ψ_2 direction from the Eguchi-Hanson metric, giving fields

$$\Phi = \frac{3}{4}\ln\left\{\frac{H^{1/3}w_2^2}{4g_2}\right\} \tag{3.91}$$

$$B_{\mu\nu} = 0 \tag{3.92}$$

$$C_{\phi_2} = a_2\cos(\theta_2) \tag{3.93}$$

$$A_{tx_1x_2} = \frac{1}{H} \tag{3.94}$$

and metric

$$\begin{aligned} ds_{10}^2 = & \frac{w_2}{2}\Big\{H^{-1/2}g_2^{-1/2}\left(-dt^2+dx_1^2+dx_2^2\right)+ \\ & +H^{1/2}g_2^{-1/2}ds^2_{TN(1)}+H^{1/2}g_2^{1/2}a_2^2\Big\{dw_2^2+\frac{w_2^2}{4g_2}\left(d\theta_2^2+\sin^2\theta_2 d\phi_2^2\right)\Big\}\Big\} \end{aligned} \tag{3.95}$$

which describes another pair (for H equal to either $H_A(r_1,r_2(w_2))$ or $H_B(r_1,r_2(w_2))$) of D2/D6 systems. The world-volume of the D6 brane transverse to D2 is just a TN_4. In the large w_2 limit, the metric (3.95), reduces to the metric (3.75) with $ds^2_{EH(1)} \to ds^2_{TN_4(1)}$, which is again a 10D locally asymptotically flat metric with the Kretchmann invariant

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{A(r_1,n_1)}{w_2^2}+\frac{B(r_1,n_1,a_2)}{w_2^6} \tag{3.96}$$

which vanishes at large w_2 and $A(r_1,n_1) = \frac{384n_1^2}{(r_1+2n_1)^6}$ and

$$B(r_1,n_1,a_2) = \frac{32(7r_1^6+84n_1r_1^5+420n_1^2r_1^4+1120n_1^3r_1^3+1680n_1^4r_1^2+1344n_1^5r_1+448n_1^6)}{(r_1+2n_1)^6a_2^4}. \tag{3.97}$$

All the components of the Riemann tensor in the orthonormal basis approach zero at $w_2 \to \infty$.

4. Taub-NUT/bolt extended solutions

We consider here solutions that are obtained from generalizations of the Taub-NUT metric. These are higher-dimensional generalizations of TN_4 and its bolt-generalizations [16, 17]. We shall describe these metrics as we encounter them in turn.

These solutions do not preserve any supersymmetry, but nevertheless exhibit interesting properties that are qualitatively similar to the previous cases. Specifically the metric function H behaves the same way near the brane core and at infinity. For each solution we again find that it is an integrated product of a decaying function and a damped oscillating function far from the brane. Near the brane core, the convolution of the two functions diverges, as for the supersymmetric cases.

We note that a complete list of possible D-brane combinations that give rise to supersymmetric solutions has been compiled [9]. These solutions are obtained by applying the T-duality method to the known supersymmetric solutions. Although some of the solutions in the following subsections would seem to be supersymmetric when comparing to this map of solutions, each metric must be checked explicitly to see if it preserves any supersymmetry. For example, in subsection 4.1, although we find that we get a $2 \perp 6(2)$ solution that is on the list, it is not supersymmetric, as is the situation for all of our solutions in this section. The other system we get is a $2 \perp 4(2)$ solution which is not on the list and of course, we also find that this solution is not supersymmetric.

4.1 Embedding of four dimensional Taub-bolt space

We consider first an M2 brane in the background of Taub-bolt space given by the following metric

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(y,r)\Big(-dt^2+dx_1^2+dx_2^2\Big)+ \\ &\quad +H^{1/3}(y,r)\Big(dy^2+y^2[d\alpha^2+\sin^2\alpha(d\beta^2+\sin^2\beta d\gamma^2)]+ds_{TB_4}^2\Big)\,. \end{aligned} \tag{4.1}$$

The four-form F has the following non-vanishing components,

$$F_{tx_1x_2y} = -\frac{1}{2H}\frac{\partial H}{\partial y} \tag{4.2}$$

$$F_{tx_1x_2r} = -\frac{1}{2H}\frac{\partial H}{\partial r} \tag{4.3}$$

and the four-dimensional Taub-bolt space is given by the metric

$$ds_{TB_4}^2 = f_{4,\mathrm{bolt}}(r)dr^2 + f_{4,\mathrm{bolt}}^{-1}(r)\left(d\psi+2n\cos\theta d\phi\right)^2+\left(r^2-n^2\right)\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{4.4}$$

where the function $f_4(r)$ is given by

$$f_{4,\mathrm{bolt}}(r) = \frac{2(r^2-n^2)}{(r-2n)(2r-n)}\,. \tag{4.5}$$

Note that we have already applied the conditions on $f_4(r)$ such that (4.4) is a bolt solution. Briefly, the difference between the NUT and bolt solutions are as follows. The metric (4.4) is obtained by fixing the mass parameter in the general metric function of the Taub-NUT space

$$f_4(r) = \frac{r^2 - n^2}{r^2 + n^2 - 2mr} \tag{4.6}$$

so that f_4^{-1} vanishes for $r = r_b = 2n > n$. This occurs if $m = \frac{5}{4}n$, and the fixed point set of the Killing vector $\partial/\partial\psi$ is a two-dimensional sphere with radius $\sqrt{3}n$. On the other hand, if we fix the mass parameter to be $m = n$, then we have a NUT solution at $r = r_n = n$, where the fixed point set of the Killing vector $\partial/\partial\psi$ is zero-dimensional. In this case, the function $f_4(r)$ reduces to $f_{4,\mathrm{NUT}}(r) = \frac{r+n}{r-n}$ (which is the case we discussed in section 3). We note that in both cases, $\frac{1}{f_{4,\mathrm{bolt}}(r)}\Big|_{r=r_b} = \frac{1}{f_{4,\mathrm{NUT}}(r)}\Big|_{r=r_n} = 0$ and $\frac{d}{dr}(\frac{1}{f_{4,\mathrm{bolt}}(r)})\Big|_{r=r_b} = \frac{d}{dr}(\frac{1}{f_{4,\mathrm{NUT}}(r)})\Big|_{r=r_n} = \frac{1}{2n}$.

An equivalent form of the metric (4.4) can be written by shifting r by a constant and scaling $\psi = 4n\Psi$, yielding

$$ds^2_{TB_4} = \widetilde{f}_{4,\mathrm{bolt}}(r)dr^2 + 16n^2\widetilde{f}^{-1}_{4,\mathrm{bolt}}(r)\left(d\Psi + \frac{1}{2}\cos\theta d\phi\right)^2 + r(r+2n)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.7}$$

where

$$\widetilde{f}_{4,\mathrm{bolt}}(r) = \frac{2r(r+2n)}{(r-n)(2r+n)} \tag{4.8}$$

and the radius R_∞ of the circle at $r \to \infty$, with line element $R^2_\infty\left(d\Psi + \frac{1}{2}\cos\theta d\phi\right)^2$, fibered over base space S^2 of the Taub-bolt space, is given by

$$R_\infty = 4n\,. \tag{4.9}$$

We note that the coordinate r in (4.7) ranges over $[n, \infty)$, the coordinates ϕ and Ψ take values on the interval $[0, 2\pi]$, and θ on the interval $[0, \pi]$.

The metric (4.1) and four-form (4.2), (4.3) satisfy the field equations of $d = 11$ supergravity if $H(y, r)$ satisfies the following equation

$$(2r^2 - rn - n^2)\frac{\partial^2 H}{\partial r^2} + (4r - n)\frac{\partial H}{\partial r} + 2r(r+2n)\frac{\partial^2 H}{\partial y^2} + \frac{6r}{y}(r+2n)\frac{\partial H}{\partial y} = 0\,. \tag{4.10}$$

By separating the coordinates using the relation

$$H(y, r) = 1 + Q_{M2}Y(y)R(r) \tag{4.11}$$

we find that the function $Y(y)$ obeys equation (3.9), and so

$$Y(y) \sim \frac{J_1(cy)}{y}\,. \tag{4.12}$$

The radial equation is given by

$$(2r^2 - n^2 - rn)R'' + (4r - n)R' - 2rc^2(r+2n)R = 0 \tag{4.13}$$

and a typical numerical solution of (4.13) versus $n/2r$ is given in the figure 4. The solution $R_c(r)$ logarithmically diverges at $r = n$.

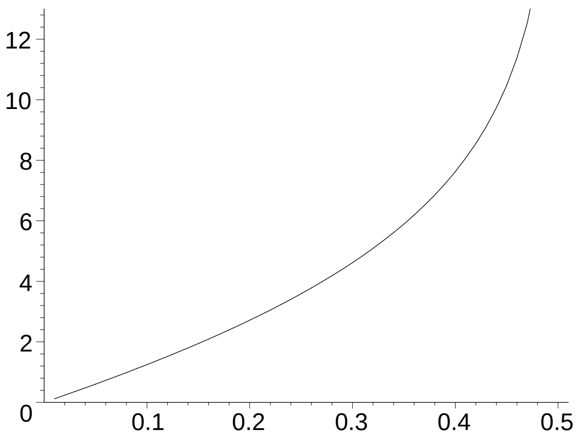

Figure 4: Numerical solution of radial equation (4.13) for $R/10^7$ in Taub-bolt$_4$ case, as a function of $n/2r$. So for $r \approx n$, R diverges and for $r \approx \infty$, it vanishes.

We note that the most general solution for the metric function is a superposition of the different functions in the solution (4.11), corresponding to different values of the separation constant c. By knowing the particular numerical solution for $R_c(r)$, the most general solution is

$$H_{TB_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc p(c) R_c(r)\frac{J_1(cy)}{y}\,. \tag{4.14}$$

By dimensional analysis, $p(c) = p_0 c^4$, where p_0 is a constant that can be absorbed into the definition of Q_{M2}. Since the graph of $R_c(r)$ given in (4) is qualitatively the same as the corresponding graph for TN$_4$ (both diverge at the brane location, and vanish at infinity), the behaviour of the metric function (4.14), is similar to the NUT case (3.10).

The other solution is obtained by changing $c \to ic$:

$$\widetilde{H}_{TB_4}(y,r) = 1 + Q_{M2}\int_0^\infty dc\widetilde{p}(c)\widetilde{R_c}(r)\frac{K_1(cy)}{y} \tag{4.15}$$

where $\widetilde{R_c}(r)$ is a damped oscillating function. A plot of this function is given in figure 5. Again by dimensional analysis, $\widetilde{p}(c) = \widetilde{p}_0 c^4$. The qualitative form of $\widetilde{H}_{TB_4}$ is similar to that of $\widetilde{H}_{TN}$ in (3.21), and a plot of $\left(\widetilde{H}_{TB_4}(y, r=0) - 1\right)$ is the same as the dotted line in figure 1.

Dimensional reduction of our bolt based solution (4.1) in the Ψ direction of (4.7) gives the type-IIA supergravity NSNS fields:

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left(\frac{\sqrt[3]{H}}{\widetilde{f}_{4,\mathrm{bolt}}(r)}\right) \\ B_{\mu v} &= 0 \end{aligned} \tag{4.16}$$

and RR fields C_μ and $A_{\mu\nu\rho}$ have the non-vanishing components:

$$\begin{aligned} C_\phi &= 2n\cos\theta \\ A_{tx_1x_2} &= H^{-1}\,. \end{aligned} \tag{4.17}$$

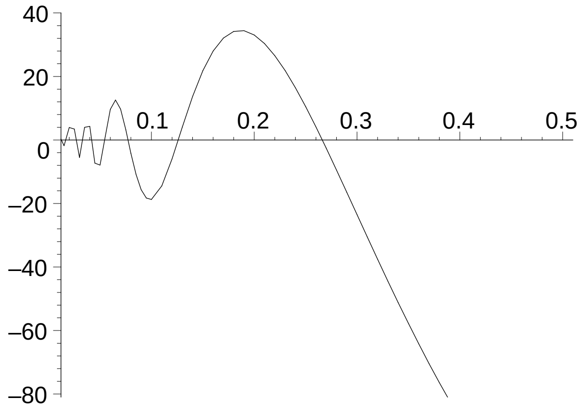

Figure 5: Numerical solution of damped oscillating radial equation for $\widetilde{R}/10$ in Taub-bolt$_4$ case, as a function of $n/2r$. So for $r \approx n$, $\widetilde{R}$ diverges and for $r \approx \infty$, it vanishes.

The string-frame 10-dim metric is given by

$$\begin{aligned} ds_{10}^2 &= \frac{H^{-1/2}}{\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}}\Big(-dt^2 + dx_1^2 + dx_2^2\Big) + \\ &\quad + \frac{H^{1/2}}{\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}}\Big(dy^2 + y^2 d\Omega_3^2 + \widetilde{f}_{4,\mathrm{bolt}} dr^2 + r(r+2n)d\Omega_2^2\Big) \\ &= \frac{H^{-1/2}}{\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}}\Big(-dt^2 + dx_1^2 + dx_2^2\Big) + \frac{H^{1/2}}{\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}}\Big(dy^2 + y^2 d\Omega_3^2\Big) + \\ &\quad + H^{1/2}\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}\Big(dr'^2 + r'(r' + \frac{3}{2}n)d\Omega_2^2\Big) \end{aligned} \tag{4.18}$$

where H is equal to either of H_{TB_4} or $\widetilde{H}_{TB_4}$, respectively describing distinct D2/D6 brane systems that overlap in two directions. The function $\widetilde{f}_{4,\mathrm{bolt}}$ is given by relation (4.8) and we have shifted the coordinate r to $r' = r - n$. If we compare our 10-dim metric with the result obtained in [5], in which the Taub-NUT based M-theory was considered, we note that the normal space to the D2/D6 system is distorted from a perfect sphere (up to the conformal factor $H^{1/2}\sqrt{\widetilde{f}_{4,\mathrm{bolt}}}$) by the NUT charge. We have explicitly checked that the above 10-dimensional metric, with the given dilaton and one form, *is* a solution to the 10-dimensional supergravity equations of motion.

4.2 Six dimensional Taub-NUT and Taub-bolt based M-theory

We now demonstrate that M2 branes can be described using higher-dimensional Taub-NUT/bolt spaces as backgrounds. Accordingly, we begin by embedding a six-dimensional Taub-NUT/bolt space into the eleven-dimensional metric, given by

$$ds_{11}^2 = H^{-2/3}(y,r)(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(y,r)(dy^2 + y^2 d\alpha^2 + ds_{TN_6}^2) \tag{4.19}$$

and four-form F with non-vanishing components

$$F_{tx_1x_2y} = -\frac{1}{2H^2}\frac{\partial H}{\partial y} \tag{4.20}$$

$$F_{tx_1x_2r} = -\frac{1}{2H^2}\frac{\partial H}{\partial r} \tag{4.21}$$

where

$$\begin{aligned} ds^2_{TN_6} &= f_6(r)dr^2 + f_6^{-1}(r)\left(d\psi + 2n\cos\theta_1 d\phi_1 + 2n\cos\theta_2 d\phi_2\right)^2 + \\ &\quad +(r^2-n^2)\left(d\theta_1^2 + \sin^2(\theta_1)d\phi_1^2 + d\theta_2^2 + \sin^2(\theta_2)d\phi_2^2\right) \end{aligned} \tag{4.22}$$

is the metric of a six dimensional space with NUT charge. The function $f(r)$ is given by

$$f_6(r) = \frac{3(r^2-n^2)^2}{r^4 - 6r^2n^2 - 6mr - 3n^4}\,. \tag{4.23}$$

If we fix the mass parameter to be $m = -4n^3/3$, then we have a NUT solution at $r = r_n = n$ and the function $f_6(r)$ reduces to

$$f_{6,\mathrm{NUT}}(r) = \frac{3(r+n)^2}{(r-n)(r+3n)}\,. \tag{4.24}$$

On the other hand, to have a bolt solution at $r = r_b > n$, we must fix the mass parameter to be $m = \frac{1}{6}(r_b^3 - 6n^2 r_b - \frac{3n^4}{r_b})$. The bolt is located at $r_b = 3n$ and the function $f_6(r)$ reduces to

$$f_{6,\mathrm{bolt}}(r) = \frac{3(r-n)^2}{(r+n)(r-3n)}\,. \tag{4.25}$$

In this case, the mass parameter is simply $m = 4n^3/3$. We note that the function $f_{6,\mathrm{bolt}}(r)$ can be obtained from $f_{6,\mathrm{NUT}}(r)$ by changing $n \to -n$. The locations of the NUT and the bolt are determined by the relation $\frac{d}{dr}(\frac{1}{f_{6,\mathrm{NUT}}(r)})\Big|_{r=r_n} = \frac{d}{dr}(\frac{1}{f_{6,\mathrm{bolt}}(r)})\Big|_{r=r_b} = \frac{1}{3n}$, which is a necessary condition for regularity of the metric, yielding a consistent period for the coordinate ψ.

The radius R_∞ of circle at $r{=}\infty$, with line element $R^2_\infty\{d\Psi + \frac{1}{3}\cos\theta_1 d\phi_1 + \frac{1}{3}\cos\theta_2 d\phi_2\}^2$, fibered over base space $S^2 \otimes S^2$ of the Taub-NUT (and Taub-bolt) space is given by

$$R_\infty = \frac{6}{\sqrt{3}}n = \mp\frac{6}{\sqrt[6]{48}}m^{1/3}\,. \tag{4.26}$$

In the above equation and in the following, the upper (lower) sign refers to Taub-NUT (bolt) space. The coordinates ϕ_1, ϕ_2 and Ψ take values on the interval $[0, 2\pi]$, while θ_1 and θ_2 change over $[0, \pi]$. For later convenience, we can use the following form for the Taub-NUT (bolt) space

$$\begin{aligned} ds^2_{TN_6\pm} &= g_{6\pm}(r)dr^2 + 36n^2 g_{6\pm}^{-1}(r)\left(d\Psi + \frac{1}{3}\cos\theta_1 d\phi_1 + \frac{1}{3}\cos\theta_2 d\phi_2\right)^2 + \\ &\quad +r(r\pm 2n)\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) \end{aligned} \tag{4.27}$$

where

$$g_{6\pm}(r) = \frac{3(r \pm 2n)^2}{r(r \pm 4n)} \tag{4.28}$$

and the coordinate r belongs to $[0, \infty)$ for Taub-NUT space and $[4n, \infty)$ for Taub-bolt space (in contrast to (4.22), where the respective minimum values of the coordinate r are n and $3n$ for the NUT/bolt solutions). The metric (4.19) and four-form (4.20), (4.21) satisfy the field equations of $d = 11$ supergravity, provided $H(y, r)$ satisfies the following Laplace equation

$$(r \pm 4n)ry\frac{\partial^2 H_\pm}{\partial r^2} + 4y(r \pm 3n)\frac{\partial H_\pm}{\partial r} + 3y(r \pm 2n)^2\frac{\partial^2 H_\pm}{\partial y^2} + 3(r \pm 2n)^2\frac{\partial H_\pm}{\partial y} = 0\,. \tag{4.29}$$

If we separate the coordinates by

$$H_\pm(y, r) = 1 + Q_{M2}Y(y)R_\pm(r) \tag{4.30}$$

then the function $Y(y)$, which must remain finite at $y = 0$, is given by

$$Y(y) \approx J_0(qy) \tag{4.31}$$

and after a change of coordinate to $t = r \pm 2n$, the function $R_\pm(t)$ should satisfy

$$(t^2 - 4n^2)R''_\pm(t) + 4(t \pm n)R'_\pm(t) - 3q^2t^2R_\pm(t) = 0\,. \tag{4.32}$$

At large distance $r \sim t \to \infty$, the solution of (4.32) decays according to:

$$R_\pm(t) \approx \frac{(3tq^2 + \sqrt{3}\,|q|)}{t^3}e^{-\sqrt{3}|q|t}\,. \tag{4.33}$$

The metric (4.27) for the NUT solution, at small radius ($r \ll n$) behaves like:

$$\begin{aligned} ds^2_{T\ll 1} &= 12n^2\left\{dT^2 + T^2\left(d\Omega_1^2 + \frac{d\Omega_2^2 + d\Omega_2'^2}{6}\right)\right\} \\ &\equiv d\mathcal{T}^2 + \mathcal{T}^2 d\Omega_5^2 \end{aligned} \tag{4.34}$$

where $T = \sqrt{(t - 2n)/n}$ and $\mathcal{T} = 2\sqrt{3}nT$. At small distances $r \ll n$ (or $\mathcal{T} \to 0$), equation (4.32) has the solution

$$R_+(\mathcal{T}) \approx \frac{1}{\mathcal{T}^2}N_2(iq\mathcal{T}) \tag{4.35}$$

for the NUT case, where $N_2(x)$ is the second order Bessel function of the second kind. The solution (4.35) diverges as $1/\mathcal{T}^4$, and decreases with increasing $\mathcal{T}$. The other solution of the equation (4.32) is finite at small distances.

The divergence of the radial function at small distances is expected (similar to the TN_4 case, which from equation (3.11) diverges for small r, as $1/r$). As in the TN_4 cases, by a suitable choice of normalization factor we can find well-defined functions (like those in figure 1), in terms of the metric functions (4.37) and (4.38), that are finite for small values of the radial coordinate r.

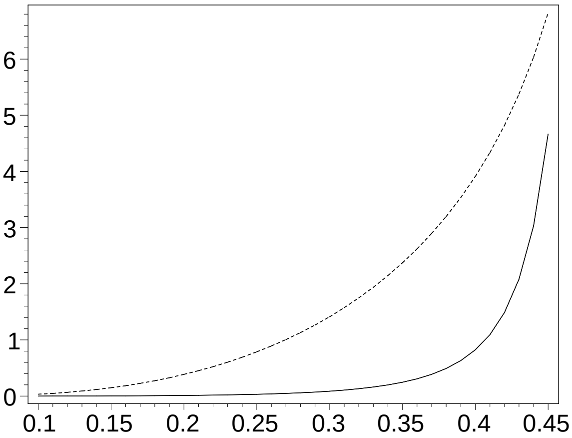

Figure 6: Numerical solutions of radial equations (4.32) for $R_+/10^5$ and $R_-/10^3$ in the Taub-NUT$_6$ (solid) and Taub-bolt$_6$ (dotted) cases, as functions of $1/t$. So for $r \approx n$ $(4n)$ (or $1/t \approx 1/2n$), $R_\pm$ diverge and for $r \approx \infty$ (or $1/t \approx 0$), they vanish.

In the case of the bolt solution, near the brane core $r \sim 4n$ (or $t \to 2n$), the equation (4.32), has the solution:

$$R_-(\mathcal{T}) \approx N_0(iq\mathcal{T}) \tag{4.36}$$

where $N_0(x)$ is the zeroth order Bessel function of the second kind. The solution (4.36) diverges on the brane as $\ln \mathcal{T}$, more softly than the NUT case, and decreases for increasing small $\mathcal{T}$. A typical numerical solution of (4.32) for NUT and bolt cases versus $1/t$ for $n = 1$ is given in the figure 6.

As before, the most general solution is of the form

$$H_{TN_6\pm}(y,r) = 1 + Q_{M2}\int_0^\infty dc s_\pm(c) R_{c\pm}(r) J_0(cy) \tag{4.37}$$

where by dimensional analysis, the measure function must have the form $s_\pm(c) = s_{0\pm}c^5$. The exact value of $s_{0\pm}$ must be obtained by looking at near horizon geometry. Two other solutions are obtained by analytic continuation of $c \to ic$, yielding

$$\widetilde{H}_{TN_6\pm}(y,r) = 1 + Q_{M2}\int_0^\infty dc \widetilde{s}_\pm(c) \widetilde{R}_{c\pm}(r) K_0(cy) \tag{4.38}$$

where the typical behaviors of the functions $\widetilde{R}_{c\pm}(r)$ are plotted in the figure 7 where we set $n = 1$.

Dimensional reduction of our solution in the Ψ direction of (4.27) gives the type-IIA supergravity NSNS fields:

$$\begin{aligned} \Phi &= \frac{3}{4}\ln\left(\frac{\sqrt[3]{H}}{g_6(r)}\right) \\ B_{\mu v} &= 0 \end{aligned} \tag{4.39}$$

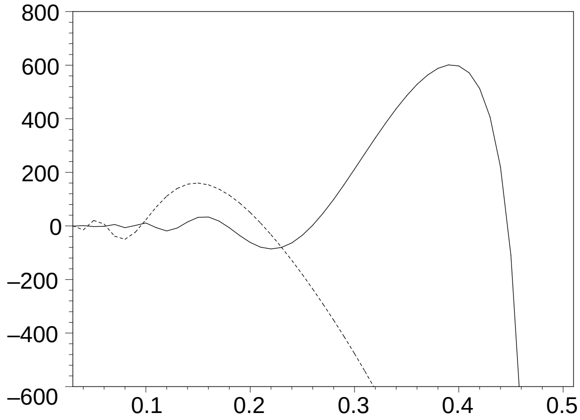

Figure 7: Numerical solutions of damped oscillating radial equations for $\widetilde{R}_+$ and $10^5\widetilde{R}_-$ in the Taub-NUT$_6$ (solid) and Taub-bolt$_6$ (dotted) cases, as functions of $1/t$. So for $r \approx n$ $(4n)$ (or $1/t \approx 1/2n$), $\widetilde{R}_\pm$ diverge and for $r \approx \infty$ (or $1/t \approx 0$), they vanish.

and RR fields C_μ and $A_{\mu\nu\rho}$ have the non-vanishing components:

$$\begin{aligned} C_{\phi_1} &= 2n\cos\theta_1 \\ C_{\phi_2} &= 2n\cos\theta_2 \\ A_{tx_1x_2} &= H^{-1}\,. \end{aligned} \tag{4.40}$$

The string-frame 10-dim metric is given by

$$\begin{aligned} ds^2_{10\pm} &= H^{-1/2}g_6^{-1/2}(-dt^2+dx_1^2+dx_2^2)+H^{1/2}g_6^{-1/2}\left(dy^2+y^2d\alpha^2\right)+ \\ &+H^{1/2}g_6^{1/2}\left(dr^2+\frac{r^2(r\pm 4n)}{3(r\pm 2n)}\left(d\Omega_2^2+d\Omega_2'^2\right)\right)\,. \end{aligned} \tag{4.41}$$

The function H is either $H_{TN_6\pm}$ or $\widetilde{H}_{TN_6\pm}$ and the function $g_6 = g_{6\pm}(r)$ given by relation (4.28). The 10-dim metric describes in each case a D2/D4 brane system, overlapping in two directions x_1 and x_2.

We have explicitly checked that the above 10-dimensional metric, with the given dilaton and one form, is a solution to the 10-dimensional supergravity equations of motion. However this solution, in the notation of [9], is $2 \perp 4(2)$ and does not preserve any supersymmetry. This latter fact could be obtained by the observation that it cannot be obtained by a T-dualization of the known supersymmetric D-brane solutions. Note that T-dualization is a powerful mathematical method for finding new supersymmetric brane solutions from known supersymmetric solutions. There is no guarantee that it gives all the supersymmetric brane solutions from a given theory.

4.3 Eight dimensional Taub-NUT and Taub-bolt based M-theory

Another case of interest is that of an M2 brane in the background of an eight-dimensional Taub-NUT/bolt space. Here we have the following metric

$$ds^2_{11} = H^{-2/3}(r)(-dt^2+dx_1^2+dx_2^2)+H^{1/3}(r)ds_8^2 \tag{4.42}$$

with the 4-form F given by the same non-vanishing components (4.20), (4.21). The eight-dimensional space with NUT charge is given by the metric

$$ds_8^2 = f_8(r)dr^2 + f_8^{-1}(r)\left(d\psi + 2n\cos\theta_1 d\phi_1 + 2n\cos\theta_2 d\phi_2 + 2n\cos\theta_3 d\phi_3\right)^2 + \\ +(r^2-n^2)\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + d\theta_3^2 + \sin^2\theta_3 d\phi_3^2\right) \tag{4.43}$$

where the function $f_8(r)$ is given by

$$f_8(r) = \frac{5(r^2-n^2)^3}{r^6 - 5r^4n^2 + 15r^2n^4 - 10mr + 5n^6}\,. \tag{4.44}$$

If we fix the mass parameter to be $m = 8n^5/5$, then we have a NUT solution at $r = r_n = n$ [17] and the function $f_8(r)$ reduces to

$$f_{8N}(r) = \frac{5(r+n)^3}{r^3 + 3nr^2 + n^2r - 5n^3}\,. \tag{4.45}$$

On the other hand, to have a bolt solution at $r = r_b > n$, we must fix the mass parameter to be $m = \frac{1}{10}(r_b^5 - 5n^2r_b^3 + 15n^4r_b + \frac{5n^6}{r_b})$. The bolt is located at $r_b = 4n$ and the function $f_8(r)$ reduces to

$$f_{8B}(r) = \frac{20(r^2-n^2)^3}{4r^6 - 20n^2r^4 + 60n^4r^2 - 3061n^5r + 20n^6} \tag{4.46}$$

where the mass parameter simplifies to $m = 3061n^5/40$.

The radius R_∞ of circle at $r = \infty$, with line element $R_\infty^2\{d\Psi + \frac{1}{4}\cos\theta_1 d\phi_1 + \frac{1}{4}\cos\theta_2 d\phi_2 + \frac{1}{4}\cos\theta_3 d\phi_3\}^2$, fibered over base space $S^2 \otimes S^2 \otimes S^2$ of the space (4.43) is given by

$$R_\infty = \frac{8}{\sqrt{5}}n \tag{4.47}$$

which for Taub-NUT space is equal to $\frac{\sqrt[5]{4096}}{\sqrt[10]{125}}m^{1/5}$ and for Taub-bolt space is equal to $\frac{8\sqrt[5]{40}}{\sqrt{5}\sqrt[5]{3061}}m^{1/5}$. The coordinates ϕ_i and Ψ take values on the interval $[0, 2\pi]$, while the θ_i range over $[0, \pi]$. For later convenience, we can use the following form for the Taub-NUT (bolt) space

$$ds_8^2 = g_8(r)dr^2 + 64n^2g_8^{-1}(r)\left\{d\Psi + \frac{1}{4}\cos\theta_1 d\phi_1 + \frac{1}{4}\cos\theta_2 d\phi_2 + \frac{1}{4}\cos\theta_3 d\phi_3\right\}^2 + \\ +r(r+2n)\left\{d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + d\theta_3^2 + \sin^2\theta_3 d\phi_3^2\right\} \tag{4.48}$$

where the function $g_8(r)$ for Taub-NUT space is

$$g_{8N}(r) = \frac{5(r+2n)^3}{r(r^2 + 6nr + 10n^2)} \tag{4.49}$$

and for Taub-bolt space is given by

$$g_{8B}(r) = \frac{5r^3(r+2n)^3}{r^6 + 6nr^5 + 10n^2r^4 - \frac{2997}{4}n^5(r+n)} \tag{4.50}$$

and the coordinate r belongs to $[0, \infty)$ for the NUT solution and to $[3n, \infty)$ for the bolt solution. Note that in (4.43), the minimum value of the coordinate r is n for the NUT solution and $4n$ for the bolt solution.

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4.3.1 Taub-NUT case

In this subsection, we consider the NUT solution. The metric (4.42) with the NUT solution metric function (4.49) and four-form (4.20), (4.21) satisfy the field equations of $d = 11$ supergravity, if $H(r)$ satisfies to the following Laplace equation

$$r(r^2+6rn+10n^2)\frac{\partial^2 H}{\partial r^2}+2(3r^2+15rn+20n^2)\frac{\partial H}{\partial r}=0 \tag{4.51}$$

which has an exact solution

$$H(r)=\frac{1}{30n^2r^3}-\frac{3}{100n^3r^2}+\frac{13}{500n^4r}+\frac{12\ln\frac{r^2}{r^2+6rn+10n^2}-7\{\tan^{-1}(\frac{r}{n}+3)-\frac{\pi}{2}\}}{2500n^5} \tag{4.52}$$

that behaves appropriately at small and large distances. For large values of r, it vanishes as

$$H(r)\sim\frac{1}{r^5} \tag{4.53}$$

and for small values of r, it diverges as

$$H(r)\sim\frac{1}{r^3}\,. \tag{4.54}$$

In this case, the dimensional reduction of our solution in the Ψ direction of (4.48) gives the type-IIA supergravity NSNS fields:

$$\begin{aligned}\Phi &= \frac{3}{4}\ln\left(\frac{\sqrt[3]{H}}{g_8(r)}\right)\\ B_{\mu\upsilon} &= 0\end{aligned} \tag{4.55}$$

and the RR fields C_μ and $A_{\mu\nu\rho}$ have the non-vanishing components:

$$\begin{aligned}C_{\phi_1} &= 2n\cos\theta_1\\ C_{\phi_2} &= 2n\cos\theta_2\\ C_{\phi_3} &= 2n\cos\theta_3\\ A_{tx_1x_2} &= H^{-1}\end{aligned} \tag{4.56}$$

The string-frame 10-dim metric

$$ds_{10}^2=\frac{H^{-1/2}}{\sqrt{g_8(r)}}(-dt^2+dx_1^2+dx_2^2)+\frac{H^{1/2}}{\sqrt{g_8(r)}}\left\{g_8(r)dr^2+r(r+2n)(d\Omega_2^2+d\Omega_2'^2+d\Omega_2''^2)\right\} \tag{4.57}$$

describes a D2 brane localized at a point of the 7-dimensional background space where the function $g_8(r)$ is given by relation (4.49). We have explicitly checked that the above 10-dimensional metric, with the given dilaton and one form, is a solution to the 10-dimensional supergravity equations of motion.

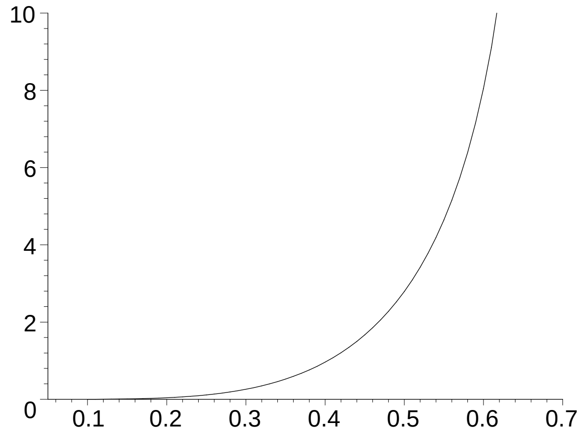

Figure 8: Numerical solution of Laplace equation (4.58) for $H/10^{25}$ in Taub-bolt$_8$, as a function of $2n/r$. So for $r \approx 3n$, H diverges and for $r \approx \infty$, vanishes.

4.3.2 Taub-bolt case

Now, we consider the Taub-bolt case. The metric (4.42) with the bolt solution metric function (4.50) and four-form (4.20), (4.21) satisfy the field equations of $d = 11$ supergravity, if $H(r)$ satisfies to the following Laplace equation

$$\{4r^6+24nr^5+40n^2r^4-2997n^5(r+n)\}H''+(24r^5+120nr^4+160n^2r^3-2997n^5)H' = 0 \quad (4.58)$$

which has the solution

$$H = \int \frac{d\widetilde{r}}{(16875n^5 + 13500\widetilde{r}n^4 + 4800\widetilde{r}^2n^3 + 940\widetilde{r}^3n^2 + 96\widetilde{r}^4n + 4\widetilde{r}^5)\widetilde{r}} \quad (4.59)$$

where we have set $r \to \widetilde{r} = r - 3n$ so that $\widetilde{r} = 0$ is the location of the bolt. The function H diverges logarithmically near the bolt and vanishes at infinity as $1/\widetilde{r}^5$. A numerical solution of the equation (4.58) is shown in the figure 8. The type-IIA supergravity NSNS, RR fields and 10-dimensional string metric have the same form as (4.55), (4.56) and (4.57) where the function $g_8(r)$ is given by relation (4.50).

5. Decoupling limits

In this section we wish to discuss the decoupling limits for the various solutions we have presented above. The specifics of calculating the decoupling limit are shown in detail elsewhere (see for example [18]), so we will only provide a brief outline here. The process is the same for all cases, so we will also only provide specific examples of a few of the solutions above.

At low energies, the dynamics of the D2 brane decouple from the bulk, with the region close to the D6 brane corresponding to a range of energy scales governed by the IR fixed point [18]. For D2 branes localized on D6 branes, this corresponds in the field theory to a vanishing mass for the fundamental hyper-multiplets. Near the D2 brane horizon ($H \gg 1$), the field theory limit is given by

$$g^2_{\mathrm{YM2}} = g_s \ell_s^{-1} = \text{fixed}\,. \quad (5.1)$$

In this limit the gauge couplings in the bulk go to zero, so the dynamics there decouple. In each of our cases above, the radial coordinates are also scaled such that

$$Y = \frac{y}{\ell_s^2}\,, \qquad U_i = \frac{r_i}{\ell_s^2} \tag{5.2}$$

are fixed (where Y and U_i, $i = 1,2$ are used where appropriate). As an example we note that this will change the harmonic function of the D6 brane in the $\mathrm{TN}_4 \otimes \mathrm{TN}_4$ case to the following (recall that the asymptotic radius of the 11th dimension is $R_\infty = 4n_2 = g_s\ell_s$)

$$f_2(r_2) = \left(1 + \frac{2n_2}{r_2}\right) = \left(1 + \frac{g_s\ell_s}{2r_2}\right) = \left(1 + \frac{g_s}{2\ell_s U_2}\right) = \left(1 + \frac{g_{\mathrm{YM2}}^2 N_6}{2U_2}\right) = f(U_2) \tag{5.3}$$

(generalizing to the case of N_6 D6 branes as was done in [5], giving the factor of N_6 in the final line above — this function is of course the same as the harmonic function of the D6 brane for the TN_4 case from [5], as is to be expected). For the Taub-NUT case, $f(U)$ is given by equation (5.3), whereas for the EH metric it is

$$g(r) \to \left(1 - \frac{A^4}{U^4}\right) = g(U) \tag{5.4}$$

where a has been rescaled to $a = A\ell_s^{-2}$.

All of the D2 harmonic functions from the above solutions (both supersymmetric and non-supersymmetric) can be shown to scale as $H(Y,U) = \ell_s^{-4}h(Y,U)$. This form causes the D2-brane to warp the ALE region and the asymptotically flat region of the D6-brane geometry. The $h(Y,U)$'s are easily calculated; as an example, the $\mathrm{TN}_4 \otimes \mathrm{TN}_4$ function is given by

$$\begin{aligned} h_{(TN)^2}(U_1,U_2) \;=\; & \frac{1}{2}\pi^2 N_2 g_{\mathrm{YM2}}^2 \int_0^\infty dP P^5 \Gamma\left(\frac{Pg_{\mathrm{YM2}}^2}{4}\right) e^{-iPU_1} \times \\ & \times \mathcal{G}_1\left(1 + iPm_1, 2, 2iPU_1\right) e^{-PU_2} \mathcal{U}\left(1 + \frac{Pg_{\mathrm{YM2}}^2}{4}, 2, 2PU_2\right) \end{aligned} \tag{5.5}$$

where the U_i in (5.2) with $i = 1$ and 2 have been used, as well as rescaling the $n_1 = m_1\ell_s^2$, $c = P/\ell_s^2$. Even when full analytic forms of $H(y,r)$ are not available, we can show that $H(Y,U) = \ell_s^{-4}h(Y,U)$ in the decoupling limit, due to the general forms of $H(y,r)$ we obtained above.

For the $H(Y,U)$ functions, we use $\ell_p = g_s^{1/3}\ell_s$ to rewrite

$$Q_{M2} = 32\pi^2 N_2 \ell_p^6 = 32\pi^2 N_2 g_{\mathrm{YM2}}^2 \ell_s^8\,. \tag{5.6}$$

The respective supersymmetric metrics are then given by appropriate insertion of either (5.3) or (5.4) along with the relevant $h(Y,U)$ into the metrics (3.22), (3.44), (3.60), (3.74). For example, for TN_4, the metric (3.22) scales as [5]:

$$\begin{aligned} \frac{ds_{10}^2}{\ell_s^2} = {} & h(Y,U)^{-1/2} f(U)^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \\ & + h(Y,U)^{1/2} f(U)^{-1/2}\left(dY^2 + Y^2 d\Omega_3^2\right) + \\ & + h(Y,U)^{1/2} f(U)^{1/2}\left(dU^2 + U^2 d\Omega_2^2\right) \end{aligned} \tag{5.7}$$

and there is only an overall normalization factor of ℓ_s^2 in the above metric. This is the expected result for a solution that is a supergravity dual of a QFT. As all of the other supersymmetric cases are qualitatively the same, we won't write them out here.

For the non-supersymmetric cases, we take the same limits (5.1), (5.2), but of course the metric functions will differ. As an example, for the 4-dimensional Taub-bolt case, this will change the harmonic function of the D6 brane to the following

$$\widetilde{f}_4(U) = \frac{2r(r+2n)}{(r-n)(2r+n)} = \frac{16U(g_{\mathrm{YM2}}^2 + 2U)}{(8U + g_{\mathrm{YM2}}^2)(4U - g_{\mathrm{YM2}}^2)} \tag{5.8}$$

where $R_\infty = 4n = g_s \ell_s$ has been used. The metric functions of the non-D2 branes in the TN_6,TB_6 and TN_8,TB_8 cases will be altered by the same steps. Since an analytic function for $R(r)$ cannot be found in any of the non-susy cases, the transformations of the D2 functions $H(y,r)$ cannot be found explicitly. However, using the general forms (4.14), (4.37) we can still show, for example, that

$$H_{TB4}(y,r) \to Q_{M2} \int \frac{dP}{\ell_s^2} p_0 \frac{P^4}{\ell_s^8} \frac{R_c(U) J_1(PY)}{Y \ell_s^2} = \frac{1}{\ell_s^4} h_{TB_4}(Y,U) \tag{5.9}$$

where (5.6), (5.2) and the rescaling $c = P\ell_s^{-2}$ have been used. The TB_4 metric will then become

$$\begin{aligned} \frac{ds^2}{\ell_s^2} = &\frac{1}{\sqrt{\widetilde{f}_4(U) h_{TB_4}(Y,U)}}(-dt^2 + dx_1^2 + dx_2^2) + \frac{\sqrt{h_{TB_4}(Y,U)}}{\sqrt{\widetilde{f}_4(U)}}(dY^2 + Y^2 d\Omega_3^2) + \\ &+ \sqrt{\widetilde{f}_4(U) h_{TB_4}(Y,U)} \left(dU^2 + U \left(U + \frac{3g_{\mathrm{YM}}^2}{8} \right) d\Omega_2^2 \right) \end{aligned} \tag{5.10}$$

and the $\mathrm{TN}_6/\mathrm{TB}_6$ metric in the decoupling limit will be given by

$$\begin{aligned} \frac{ds_{10}^2}{\ell_s^2} = &\frac{1}{\sqrt{h_{TN_6}(Y,U) g_6(U)}} \left(-dt^2 + dx_1^2 + dx_2^2\right) + \sqrt{\frac{h_{TN_6}(Y,U)}{g_6(U)}} \left(dY^2 + Y^2 d\alpha^2\right) + \\ &+ \sqrt{h_{TN_6}(Y,U) g_6(U)} \left(dU^2 + \frac{U^2(\sqrt{3}U \pm 2g_{\mathrm{YM}}^2)}{3(\sqrt{3}U \pm g_{\mathrm{YM}}^2)} \left(d\Omega^2 + d\Omega'^2\right) \right) \end{aligned} \tag{5.11}$$

where again we note that the only dependence on the string scale ℓ_s is in an overall normalization factor. There is of course no decoupling occurring in the eight dimensional Taub-NUT/bolt cases.

6. Conclusions

By embedding different possible spaces with NUT charge and/or self-dual curvature, we have found a rich and interesting new class of brane solutions to $D = 11$ supergravity. These exact solutions are new M2 brane metrics, and are presented in equations (3.16), (3.40), (3.41), (3.53), (3.73), (3.83), (3.85), (4.14), (4.15), (4.37), (4.38), (4.52) and (4.59) which are the main results of this paper. The common feature of all of these solutions is that the brane function is a convolution of an exponentially decaying 'radial' function with a damped oscillating one. The 'radial' functions vanish far from the M2 brane and diverge near the brane core.

Dimensional reduction to 10 dimensions gives us different D2/D6 systems (with metric functions) which in all the cases with a combination of 4 dimensional Taub-NUT and Eguchi-Hanson spaces (which have self-dual Riemann curvature), the configurations preserve 1/4 of the supersymmetry and yield metrics with acceptable asymptotic behaviour. In all other cases — involving 4 dimensional Taub-bolt and higher dimensional Taub-NUT/bolt — the D-brane system is not supersymmetric. However the general functional structure of the branes is qualitatively the same as for the supersymmetric cases: the r-dependent parts of the metric functions diverge for small r and fall off rapidly for large r, whereas the y-dependent parts of the metric functions approach a finite value for small y and vanish at large y.

Finally we considered the decoupling limit of our solutions. In the case of embedded TN_4, when the D2 brane decouples from the bulk, the theory on the brane is 3 dimensional $\mathcal{N} = 4$ $\mathrm{SU}(\mathrm{N}_2)$ super Yang-Mills (with eight supersymmetries) coupled to N_6 massless hypermultiplets [19]. This point is obtained from dual field theory and since some of our solutions preserve the same amount of supersymmetry, a similar dual field description should be attainable.

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A. Supersymmetries of the TN and EH solutions

We demonstrate in this appendix the supersymmetry of our solutions that embed one (or both) of TN_4 and EH.

We begin by noting another useful representation of the algebra (2.14), given by the following 32 dimensional representation [20]

$$\begin{aligned}
\Gamma_i &= \begin{bmatrix} 0 & -\widetilde{\Gamma}_i \\ \widetilde{\Gamma}_i & 0 \end{bmatrix} \qquad (i = 1, \ldots, 8) \\
\Gamma_9 &= \begin{bmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{bmatrix} \\
\Gamma_\# &= \begin{bmatrix} 0 & 1_{16} \\ 1_{16} & 0 \end{bmatrix} \\
\Gamma_0 &= -\Gamma_{123456789\#}
\end{aligned} \tag{A.1}$$

where $\widetilde{\Gamma}_i$, the 16 dimensional representation of the Clifford algebra in eight dimension, are given by

$$\widetilde{\Gamma}_i = \begin{bmatrix} 0 & L_i \\ L_i & 0 \end{bmatrix} \qquad (i = 1, \ldots, 7) \tag{A.2}$$

$$\widetilde{\Gamma_8} = \begin{bmatrix} 0 & -1_8 \\ 1_8 & 0 \end{bmatrix} \tag{A.3}$$

in terms of L_i, the left multiplication by the imaginary octonions on the octonions. The imaginary unit octonions satisfy the following relationship

$$e_i \cdot e_j = -\delta_{ij} + c_{ijk} e_k \tag{A.4}$$

where c_{ijk} is totally skew symmetric and its non-vanishing components are given by

$$c_{124} = c_{137} = c_{156} = c_{235} = c_{267} = c_{346} = c_{457} = 1 \tag{A.5}$$

We take the L_i to be the matrices such that the relation (A.4) holds. In other words, given a vector $v = (v_0, v_i)$ in $\mathbb{R}^8$, we write $\hat{v} = v_0 + v_j e_j$, where the effect of left multiplication is $e_i(\hat{v}) = v_0 e_i - v_i + c_{ijk} v_j e_k$, we then construct the 8×8 matrix $(L_i)_{AB}$ by requiring $e_i(\hat{v}) = (L_i)_{AB}\, e_A v_B$, where $A, B = 0, 1, \ldots, 7$. With this in mind, we obtain considerable simplification of the Killing spinor equation (2.8).

Two relations that are useful in solving for the Killing spinors are

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \tag{A.6}$$

$$\left[\Gamma^{ab}, \Gamma^{cd}\right] = 2\eta^{ad}\Gamma^{bc} + 2\eta^{bc}\Gamma^{ad} - 2\eta^{ac}\Gamma^{bd} - 2\eta^{bd}\Gamma^{ac}\,. \tag{A.7}$$

Eq. (A.6) is of course the Baker-Campbell-Hausdorff identity, and (A.7) is derivable from (2.14) and (2.13), for $p = 2$.

Consider first all of our supersymmetry preserving solutions. Using (2.8), one finds

$$\left(1 + \Gamma^{\hat{t}\hat{x}_1\hat{x}_2}\right)\epsilon = 0 \tag{A.8}$$

(where hats over coordinates denote tangent space indices), and so at most half the supersymmetry is preserved due to the presence of the brane.

Now, consider an embedding of TN_4. The remaining equations from (2.8), arising from the left-over terms from $\partial_M\epsilon + \frac{1}{4}\omega_{Mab}\Gamma^{ab}\epsilon$ portion, are

$$\partial_{\alpha_1}\epsilon - \frac{1}{2}\Gamma^{\hat{y}\hat{\alpha}_1}\epsilon = 0 \tag{A.9}$$

$$\partial_{\alpha_2}\epsilon - \frac{\sin(\alpha_1)}{2}\Gamma^{\hat{y}\hat{\alpha}_2}\epsilon - \frac{\cos(\alpha_1)}{2}\Gamma^{\hat{\alpha}_1\hat{\alpha}_2}\epsilon = 0 \tag{A.10}$$

$$\partial_{\alpha_3}\epsilon - \frac{\sin(\alpha_1)\sin(\alpha_2)}{2}\Gamma^{\hat{y}\hat{\alpha}_3}\epsilon - \frac{\sin(\alpha_2)\cos(\alpha_1)}{2}\Gamma^{\hat{\alpha}_1\hat{\alpha}_3}\epsilon - \frac{\cos(\alpha_2)}{2}\Gamma^{\hat{\alpha}_2\hat{\alpha}_3}\epsilon = 0 \tag{A.11}$$

$$\partial_\psi\epsilon + \frac{n}{2(r+n)^2}\left[\Gamma^{\hat{\psi}\hat{r}} + \Gamma^{\hat{\theta}\hat{\phi}}\right]\epsilon = 0 \tag{A.12}$$

$$\partial_\theta\epsilon + \frac{n}{2(r+n)}\Gamma^{\hat{\psi}\hat{\phi}}\epsilon - \frac{r}{2(r+n)}\Gamma^{\hat{r}\hat{\theta}}\epsilon = 0 \tag{A.13}$$

$$\partial_\phi\epsilon + \frac{n^2\cos(\theta)}{(r+n)^2}\left[\Gamma^{\hat{\psi}\hat{r}} + \Gamma^{\hat{\theta}\hat{\phi}}\right]\epsilon - \frac{n\sin(\theta)}{2(r+n)}\Gamma^{\hat{\psi}\hat{\theta}}\epsilon - \frac{r\sin(\theta)}{2(r+n)}\Gamma^{\hat{r}\hat{\phi}}\epsilon - \frac{\cos(\theta)}{2}\Gamma^{\hat{\theta}\hat{\phi}}\epsilon = 0\,. \tag{A.14}$$

Eqs. (A.9), (A.10) and (A.11) are solvable by use of (A.7) and (A.6) to give the following Lorentz rotation

$$\epsilon = \exp\left\{\frac{\alpha_1}{2}\Gamma^{\hat{y}\hat{\alpha}_1}\right\}\exp\left\{\frac{\alpha_2}{2}\Gamma^{\hat{\alpha}_1\hat{\alpha}_2}\right\}\exp\left\{\frac{\alpha_3}{2}\Gamma^{\hat{\alpha}_2\hat{\alpha}_3}\right\}\tilde{\epsilon}\,. \tag{A.15}$$

Now note that (A.12) can be solved for by using the projection operator

$$\Gamma^{\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\epsilon = \epsilon \tag{A.16}$$

eliminating another half of the supersymmetry provided ϵ is independent of ψ. With this projection operator, (A.13) and (A.14) can be solved, giving

$$\epsilon = \exp\left\{-\frac{\theta}{2}\Gamma^{\hat{\psi}\hat{\phi}}\right\}\exp\left\{\frac{\phi}{2}\Gamma^{\hat{\theta}\hat{\phi}}\right\}\tilde{\epsilon}\,. \tag{A.17}$$

Turning next to the EH metric, the remaining terms from the Killing spinor equation (2.8) give (A.9), (A.10) and (A.11), which have the same solution as above, and three more equations

$$\partial_\psi\epsilon + \frac{(r^4+a^4)}{4r^4}\Gamma^{\hat{\psi}\hat{r}}\epsilon + \frac{(r^4-a^4)}{4r^4}\Gamma^{\hat{\theta}\hat{\phi}}\epsilon = 0 \tag{A.18}$$

$$\partial_\theta\epsilon - \frac{1}{4g^{1/2}}\Gamma^{\hat{r}\hat{\theta}}\epsilon + \frac{1}{4g^{1/2}}\Gamma^{\hat{\psi}\hat{\phi}}\epsilon = 0 \tag{A.19}$$

$$\begin{aligned}\partial_\phi\epsilon - \frac{\sin(\theta)}{4g^{1/2}}\Gamma^{\hat{\psi}\hat{\theta}}\epsilon - \frac{\sin(\theta)}{4g^{1/2}}\Gamma^{\hat{r}\hat{\phi}}\epsilon + \frac{a^4\cos(\theta)}{2r^4}\Gamma^{\hat{\psi}\hat{r}}\epsilon + \frac{\cos(\theta)}{4g}\Gamma^{\hat{\psi}\hat{r}}\epsilon - &\\ - \frac{\cos(\theta)}{2}\Gamma^{\hat{\theta}\hat{\phi}}\epsilon + \frac{\cos(\theta)}{4g}\Gamma^{\hat{\theta}\hat{\phi}}\epsilon = 0\,.&\end{aligned} \tag{A.20}$$

Through the use of the projection operator (eliminating a further half of the supersymmetry)

$$\Gamma^{\hat{\psi}\hat{r}\hat{\theta}\hat{\phi}}\epsilon = -\epsilon \tag{A.21}$$

it can be shown that all of the terms in equations (A.19) and (A.20) are satisfied, and so $\epsilon \neq \epsilon(\theta,\phi)$, and no Lorentz rotation is necessary for these. Equation (A.18) can be solved with this projection operator giving the Lorentz rotated ϵ

$$\epsilon = \exp\left\{-\frac{\psi}{2}\Gamma^{\hat{\psi}\hat{r}}\right\}\tilde{\epsilon}\,. \tag{A.22}$$

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